The Existence of an Optimal Allocation Rate in a Dynamic Portfolio: Rebalancing

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Abstract

Rebalancing is based on the notion that the initially determined weights in a portfolio keep constant in future terms. Focusing on a portfolio consisting of a risky asset and a risk-free asset, and assuming that all funds from the portfolio are invested in every term, investors can choose an optimal weight, which decides the proportion of investment between the two assets in every term. The weight functions as an indicator of the investment ratio between the two assets, which results in automatically inducing investors to buy less when the price of the risk assets is high, and to buy more when it is low. However, this notion is based on the existence of an optimal weight. A risky asset is supposed to follow two events: up and down. These phenomena have a certain probability. The growth rate of the portfolio follows Bernoulli trials. Investors choose the ratio to maximize the expected value of the growth rate, which is a function with three arguments, namely the probability, the up ratio, and the down ratio. The optimal weight depends on these arguments. The combination among the three can influence the existence of the optimal weight. Still, we may ask, does an optimal weight exist? We derive a conclusion: the ratios of up and down respectively have an upper boundary and a lower one. This relationship prevails regardless of probability.

JEL Classification: G-11; G-12; G-32.

Keywords: Bernoulli Trials, Existence of An Optimal Weight, Investment Decisions, Portfolio Choice.

1. Introduction

Portfolio selection, as established by Markovich (1952) in a single period, determines the optimal allocation ratios of assets, based on expected returns.

However, investments are made during multiple periods. So single-period models have been modified and developed to reflect this fact. The question then arises as to whether the ratios will stray from the optimal ones determined in an initial period or not. We can respond to this issue from two viewpoints: one is that investors recalculate the ratios every period, according to changes in the market prices; the other that investors keep the initial ratios constant without recalculations. Investors do not intend to rebalance the allocation rates which fluctuate; to keep them constant over time is enough.

In line with the notion of keeping ratios constant, we can refer to Merton (1969), in which rebalancing is implicitly introduced. The dynamic framework is as follows: multiple assets, which are a source of consumption and investment, are assumed; the optimal allocation ratios are changeable during multiple periods; the return rate of each asset is randomly determined. Merton (1969) then deals with a two-asset problem, namely a risky asset and a safe asset. Investors aim to maximize their expected utilities over time under the budget constraint, which appears as a stochastic differential equation. Assuming a constant relative risk aversion type utility function, Merton (1969) deduces a constant allocation ratio of investment between the two assets regardless of time and the value of a portfolio. The optimal ratio appears as a ratio of two elements: the difference between the mean return rate of a risky asset and the rate of a safe asset, and the variance of a risky asset and the coefficient of relative risk aversion. These deterministic concepts are given in the deduction processes. Heaton and Lucas (2000) emphasizes this consistency as independent of levels of wealth and of decisions about consumption, but we consider this as rebalancing, i.e., as an advantageous strategy.

However, Morton's logic is complicated and not easy to understand the economic implication. So, we need to elucidate the rebalancing based on a simpler model, ignoring transaction costs as supposed in Glen (2011), and considering a secure asset as money.

Since rebalancing is a matter of dynamics, the methodology to describe fluctuations is crucial. In our model, fluctuations of a risky asset are expressed in terms of the Bernoulli process, while Merton (1969) postulates the Wiener process. The

Wiener process is based on a normal distribution, which is adequate in the case of large trials. But the Bernoulli process assumes a simple probability distribution. In this situation, we demonstrate how, over time, maintaining a ratio of investment between a risky asset and a safe one leads to incremental increases in the portfolio values, and how sufficient conditions for the existence of the ratio are shown in terms of combinations among upturn and downturn rates in risky asset values and the occurrence probability in fluctuations of a risky asset. Without examinations of these conditions, the optimal allocation rate theory could not exist.

Our analytical structures are organized as follows: Section 2 presents a base-line model, in which the sufficient conditions are demonstrated. Section 3 is devoted to implications of the conditions, and section 4 to numerical explanations of the conditions. In what follows, section 5 is devoted to supplementary discussions, section 6 to structural analyses of rebalancing, and section 7 to conclusions.

2. Baseline model

We assume the following: a portfolio is composed of a risky asset and money; the whole value of a portfolio in period t is used for investments and money in the t+1 period; the ratio of the two stays constant over repeated investments; the change rates of risky asset values randomly fluctuate, which comprise two values, $\theta^u > 1$ and $0 < \theta^d < 1$. Additionally, the random variables have probability p and $1 - p$, respectively. In short, the risky asset price goes up or down. Moreover, θ_t has a similar probability density and is independent over time.

We establish a system such that

$$X_t = (1 - c)\theta_t X_{t-1} + cX_{t-1}, \quad t = 1,2, \cdots \quad (1)$$

Notations are denoted below.
$X_t =$ the value of the portfolio in t period; $c =$ the ratio of money in t period to the value of the portfolio in $t - 1$ period, and $0 < c < 1$; θ_t = fluctuation rates of a risky asset in t period; its probability density function f is defined as

$$f(\theta^u) = p \text{ and } f(\theta^d) = 1 - p.$$

In addition, we can define money value in t period S_t as follows:

$$S_t = c\, X_{t-1}.$$

Furthermore, investment in a risky asset in t period is shown as

$$I_t = (1 - c)X_{t-1}.$$

R_t shows the outcome of the investment in t period; as a result, the outcomes are expressed as $R_t = I_t\,\theta_t$.
Additionally,

$$X_t = R_t + S_t \text{ and } X_{t-1} = S_t + I_t$$

prevail.

2.1 Movement of X_t

X_t is a random variable controlled by θ_t.[1] We obtain the anticipated expectation and the standard deviation of X_t as

$$E(X_t) = (1 - c)X_{t-1}E(\theta_t) + cX_{t-1}, \qquad (2)$$

$$\sigma = (1 - c)X_{t-1}\sqrt{\mathrm{Var}(\theta_t)}. \qquad (3)$$

Here,

$$E(\theta) = p\theta^u + (1 - p)\theta^d,$$

$$\sigma^2 = p(1 - p)(\theta^u - \theta^d)^2 .$$

Based on (2) and (3), we obtain a relationship between the above properties on a $\sigma - E(X_t)$ plane. This relationship, which shows an efficient frontier, is expressed as

$$E(X_t) = X_{t-1} + \frac{E(\theta_t) - 1}{\sqrt{\mathrm{Var}(\theta_t)}}\,\sigma. \qquad (4)$$

(4) implies that $E(X_t)$ increases when σ does, and vice versa, if $E(\theta_t) -$

[1] Below, the notation t can be omitted, if necessary.

$1 > 0$. In fact, this condition is satisfied, because it implies that an anticipated expectation of a change rate in a risky asset is assumed to be greater than that for a non-risky asset, specifically in terms of money, that is, 1. In short, we suppose that a risk premium naturally exists. (4) prevails concerning the anticipated expectation of X_t, but does not regarding individual X_t. In brief, X_t can go down when X_{t-1} goes up.

We notice that (4) shows the linear relationship on a $\sigma - E(X_t)$ plane, which derives from (2) and (3), such that

$$\frac{\mathrm{dE}(X_t)}{\mathrm{d}\sigma} = \frac{d\mathrm{E}(X_t)}{dc}\frac{dc}{\mathrm{d}\sigma} = \frac{(1-E(\theta))X_{t-1}}{-X_{t-1}\sqrt{\mathrm{Var}(\theta)}} = \frac{E(\theta)-1}{\sqrt{\mathrm{Var}(\theta)}} > 0.$$

In this context, the expected return rate of a risky asset is greater than that of money. This relative connection is crucial as the driving force of this mechanism, which is attained by keeping c constant over time. Without borrowing, this mechanism self-sufficiently produces a growing portfolio.

2.2 Determination of c

The next issue is the determination of c on the efficient frontier. Based on (1), we know the change rate of X_t such that

$$\frac{X_t}{x_{t-1}} = (1-\mathrm{c})\theta_t + c,$$

where the rates can be greater or less than 1 on the assumption of $0 < c < 1$. From the above,

$$\frac{1}{k}\log\left(\frac{X_k}{X_0}\right) = \frac{1}{k}\sum_{i=1}^{k} log\left((1-c)\theta_i + c\right) \qquad (5)$$

prevails. The right side in (5) shows a summation of stochastic variables with a similar probability density. So, in this context, by the law of large numbers the left side in (5) converges to the expectation of the right side in (5), when $k \to \infty$. Denoting this expectation as h, we know

$$h = plog[(c + (1-c)\theta^u] + (1-p)\log[c + (1-c)\theta^d].$$

We assume that investors intend to maximize the expected mean of $\frac{x_t}{x_{t-1}}$ and to choose c to maximize h. This incentive is upheld by market makers, rather than by end investors, who aim to maximize their utilities over a lifetime, as Merton (1969) postulates.

2.4 Sufficient conditions for the existence of c

Our purpose is to seek conditions for the existence of c, because c needs to satisfy $0 < c < 1$, and because the conditions are related to combinations among θ^u, θ^d and p, which are given based on anticipations.

At this stage, we consider (6) as a function with an argument c,

$$h(c) = (1-p)\log\left((1-c)\theta^d + c\right) + p\log\left((1-c)\theta^u + c\right). \quad (6)$$

(6) has important properties such as

$$h(0) = (1-p)\log(\theta^d) + p\log(\theta^u) \text{ and}$$

$$h(1) = 0.$$

If $h(0) \geq 0$ and $h'(0) > 0$, we can suppose that $h(c)$ can be a bell curve regarding c, due to $h(1) = 0$.

Based on this logic, we reach a conclusion that $0 < c^* < 1$ necessarily exists[2] if the following are satisfied:

Condition A:

$$(1-p)\log(\theta^d) + p\log(\theta^u) \geq 0.$$

Condition B:

$$\left.\frac{\mathrm{Dh}}{\mathrm{dc}}\right|_{c=0} > 0.$$

.

We can respectively transform the above two conditions below. From condition A,

$$(\theta^d)^{1-p}(\theta^u)^p \geq 1. \quad (7)$$

From condition B,

[2] c^* shows the value which maximizes h(c).

$p(\theta^d - \theta^u) + (1 - \theta^d)\theta^u > 0,$ which can be further transformed as

$$\frac{\theta^u(1-\theta^d)}{\theta^u-\theta^d} > \text{p}. \qquad (8)$$

Here, we search for conditions for the coincidence of (7) and (8). First, we focus on a relationship between θ^u and θ^d, keeping p constant.

We introduce $\rho = \theta^d / \theta^u$, which shows the proportion between the down rate and the up rate, and 0< ρ < 1 prevails because θ^u>1 and $0 < \theta^d < 1$.

Then, we transform (7) and (8) into $\rho - \theta^u$ planes and $\rho - \theta^d$ planes.

Second, we deduce the relationship between (7) and (8) respectively, described in terms of ρ and θ^d as

$$\theta^d \geq \rho^p, \qquad (9)$$

$$\theta^d < (1\text{-}p) + p\rho. \qquad (10)$$

Additionally, since ρ^p < (1-p) + p, θ^d is included in the region covered in (9) and (10). Third, similarly, the relationship between ρ and θ^u is expressed as

$$\theta^u \geq \rho^{-(1-p)}, \qquad (11)$$

$$\theta^u < \frac{1-p}{\rho} + p. \qquad (12)$$

θ^u is also included in the region covered in (11) and (12), because $\rho^{-(1-p)} < \frac{1-p}{\rho}$+p.

Moreover, in (9) ~ (12), at ρ =1, $\theta^u = \theta^d$ = 1.

Here, we show the above situations in Figure 1.

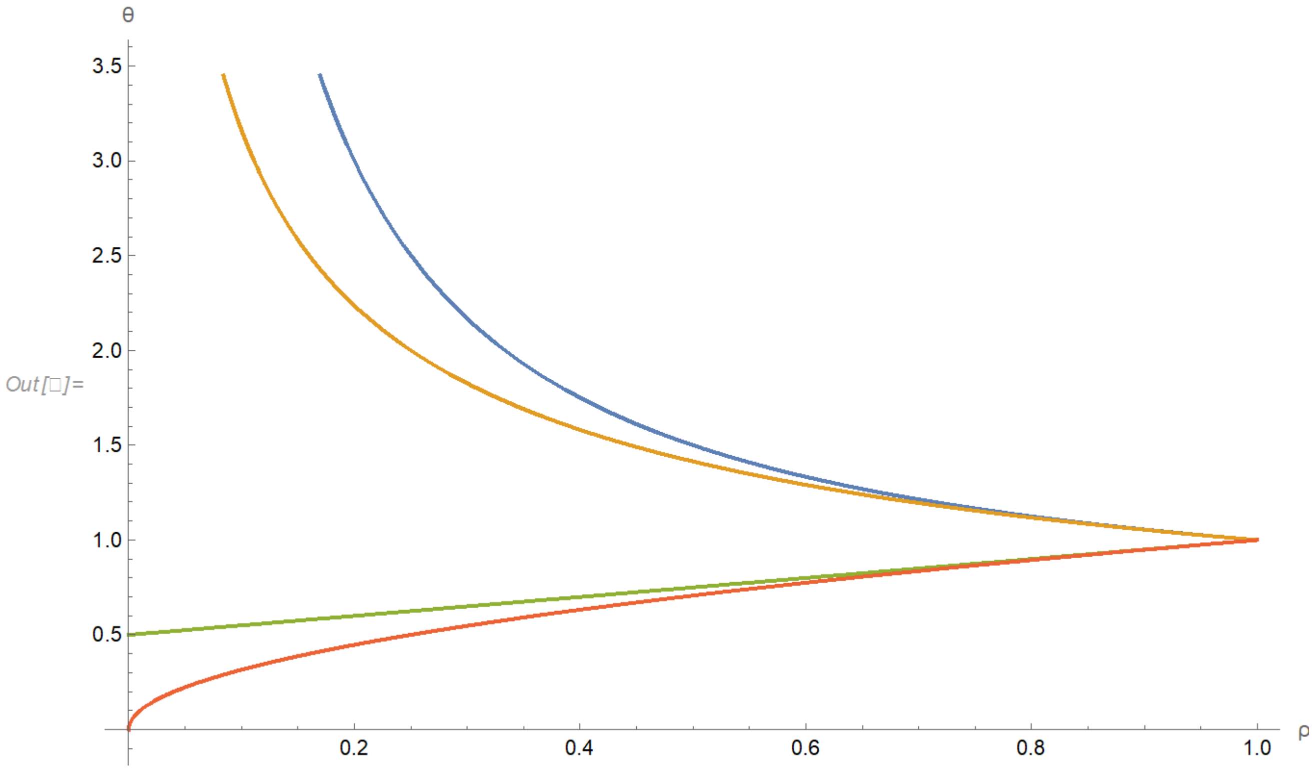

Figure 1: Regions of θs satisfying the existence of c^*.

Let us explore Figure 1 in detail: along with the vertical line, we see θ^u and θ^d, while ρ is placed along the horizontal line, and p is given;
the regions covered with colors respectively show acceptable zones of θ^u and θ^d, which satisfy (9) ~ (12), corresponding to ρ; ρ can be arbitrary in a region $0 < \rho < 1$, but we can see that there are upper boundaries and lower ones regarding θ^u and θ^d; these boundaries are determined exclusively by p and ρ, such that

$$\theta_h^u = \frac{1-p}{\rho} + \text{p}, \quad \theta_l^u = \rho^{-(1-p)}, \quad \theta_h^d = \text{p}\ \rho + (1 - \text{p}) \text{ and } \quad \theta_l^d = \rho^p.$$

Here, θ_h^u shows an upper boundary in the up rate, θ_l^u a lower one. θ_h^d shows an upper boundary in the down rate, θ_l^d a lower boundary.

3. Implications of the boundaries

In brief, regarding arbitrary ρ, θ^u and θ^d must be between the two boundaries, if c* arbitrarily exits. Figure 1 shows that investors can accept any values of ρ satisfying $0 < \rho < 1$, but the individual values of θ^u and θ^d

according to the value of ρ should be respectively accepted within the two boundaries. Otherwise, the conditions of (9) ~ (12) are partly violated, which leads to the non-existence of c*, because h(c) loses its property as a bell curve. Therefore, to keep c constant over time and to choose c in order to maximize $\frac{X_t}{X_{t-1}}$ become pointless. That is, without the above logic, the rebalancing theory is meaningless.

Furthermore, if p changes, the place relationship of the regions covered with colors remains constant. However, c^* is influenced by p, such that if p increases, c^* decreases. This shows that a high probability of upturn yields a high ratio in investments in risky assets. In short, when investors can anticipate high returns from risky assets, they prefer investments in those assets, resulting in a high ratio of risky assets.

We can verify this as follows: c^* is denoted as

$$\frac{p(\theta^u-\theta^d)+(\theta^d-1)\theta^u}{(\theta^d-1)(\theta^u-1)}.$$

Differentiating the above expression by p, the following prevails:

$$\frac{\theta^u-\theta^d}{(\theta^d-1)(\theta^u-1)}=\frac{\theta^u\left(1-\frac{\theta^d}{\theta^u}\right)}{(1-\theta^d)(1-\theta^u)}<0.$$

4. Numerical examples

Next, let us confirm this logic with numerical examples.

We assume that $\rho=0.25$ and $\mathrm{p}=0.5$. With these assumptions, $\theta_h^u=2.5$, $\theta_l^u=2$, $\theta_h^d=0.625$ and $\theta_l^d=0.5$ are determined.

Thus, we can draw up Table 1.

The anticipated values regarding θ^u and θ^d and the two tests.

	θ^u	θ^d	h(0)	h'(0)

A-1	2	0.5	$= 0$	> 0
A-2	2.5	0.625	> 0	$= 0$
A-3	2.2	0.55	> 0	> 0
A-4	3	0.75	> 0	< 0
A-5	1.2	0.3	< 0	> 0

Table 1

Here, we suppose that investors anticipate the situations from A-1 through A-5, whose anticipated rates are denoted in the 2nd and 3rd columns. These anticipated rates are used for a reference when investors decide an optimal c. In the 4th column the outcomes based on condition A are denoted, and in the 5th those based on condition B, respectively.

We specifically explain A-1and A-3: first, A-1:when investors anticipate θ^u =2 and θ^d =0.5, the places concerning θ^u and θ^d are respectively on the lower boundaries, that is, θ_l^u and θ_l^d, which satisfy the conditions, $h(0) = 0,\ h'(0) > 0;$ second, A-3: when investors anticipate θ^u =2.2 and θ^d =0.55, the places concerning θ^u and θ^d are within acceptable zones, which results in h(0) > 0 and h'(0) > 0.

As a result, A-1 and A-3 are obviously located within the acceptable zone. Concretely, in A-1, $c^* = 0.5$ and h(c^*) $= 0.059$, while in A-3, $c^* = 0.36$ and h(c^*) $= 0.115$. A-3 is more efficient than A-1. On the other hand, the other cases do not satisfy either h(0) $\geq$ 0 or h'(0) > 0, so in A-2, A-4, and A-5, c^* does not exist.

We can confirm this in Figure 2.

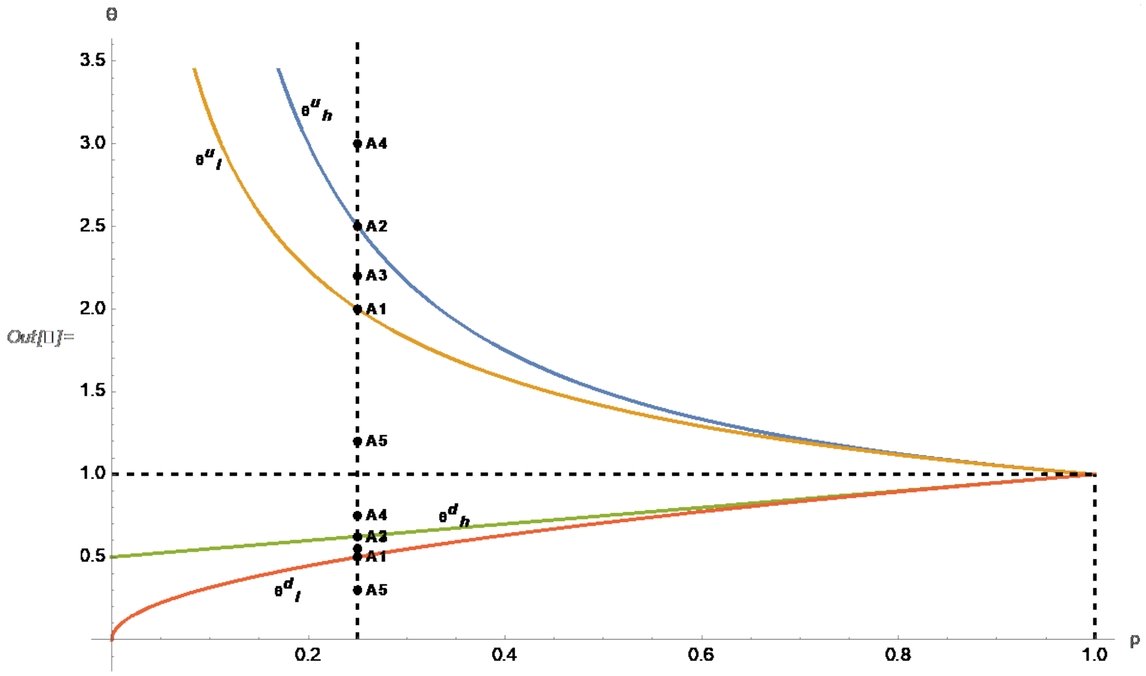

Figure 2: Locations of several situations regarding θ^u and θ^d.

In Figure 2, locations appear in pairs being separated between $\theta^u > 1$ and $\theta^d < 1$.

Luenberger (2014) cites the same situation as we show in A-1.

5. Supplementary discussions

In **2-1** we showed that the expected return rate of a risky asset is greater than that of money. What does this proposition mean in terms of the $\rho - \theta$ plane? We can express this proposition as

$$p\theta^u + (1-p)\theta^d > 1. \qquad (13)$$

First, we consider this in the $\rho - \theta^u$ plane. So, from (13) we obtain

$$\theta^u > \frac{1}{(1-p)\rho+p}.$$

Furthermore, we define a function such that

$$H(\rho) = \rho^{-(1-p)} - \frac{1}{(1-p)\rho+p},$$

which has properties such as $H(0) \to \infty$ and $H(1) = 0$. Moreover,

$$\frac{dH}{d\rho} = \frac{1-p}{\left((1-p)\rho+p\right)^2}\left(1 - \rho^{p-2}\left((1-p)\rho + p\right)^2\right) < 0,$$

because $\rho^{p-2}\left((1-p)\rho + p\right)^2$ can be transformed to $\Phi(\rho) = \left(\frac{1}{\rho}\right)^{2-p}\left((1-p)\rho + p\right)^2$, which has properties such as $\Phi(0) \to \infty$ and $\Phi(1) = 1$. As a result, $1 < \Phi(\rho)$. We can conclude that $\frac{1}{(1-p)\rho+p} < \rho^{-(1-p)}$, which supports the logic discussed in **2.4**.

Second, we proceed to analyze the $\rho - \theta^d$ plane. Based on (13),

$$\theta^d > \frac{1}{\frac{p}{\rho}+(1-p)},$$

whose right side can be transformed to $\psi(\rho) = \frac{\rho}{(1-p)\rho+p}$, which has properties such as $\psi(0) = 0$ and $\psi(1) = 1$. Then, we consider the following function as

$$\Psi(\rho) = \rho^p - \frac{\rho}{(1-p)\rho+p},$$

which features $\Psi(0) = 0$ and $\Psi(1) = 0$. Next,

$$\frac{d\Psi}{d\rho} = \frac{p(\rho^{p-1}(p+(1-p)\rho)^2-1)}{(p+(1-p)\rho)^2}$$

holds, where $\left.\frac{d\Psi}{d\rho}\right|_{\rho=0} \to \infty$. In this situation, we confirm that $\Psi(\rho) > 0$, for $0 < \rho < 1$. Since $\frac{\rho}{(1-p)\rho+p} < \rho^p$, our discussion in **2.4** is confirmed.

6. Advantages of rebalancing

The conditions discussed above give $0 < c < 1$, resulting in higher expected returns. Let us confirm this. First, consider $c = 0$, which shows the portfolio consisting exclusively of risky assets. This portfolio refers to an expected value as $(1-p)\log\theta^d + p\log\theta^u$. Second, consider the portfolio consisting of risky assets and money, whose expected values are expressed as

$$h(c) = (1-p)\log\left((1-c)\theta^d + c\right) + p\log\left((1-c)\theta^u + c\right).$$

We have supposed $h(c)$ as a bell curve regarding c. In this situation, the expected value of the portfolio consisting of risky assets appears at the origin of the bell curve. We need to confirm this logic. $h(0)$ shows the expected value of the portfolio consisting of risky assets, and this value is assumed to be the same as or greater than that of the portfolio consisting of no-risk assets, 0 in logarithmic terms.

On the other hand, the expected value of the portfolio consisting of both assets is valued at c^*, which is greater than 0. The height corresponding to c^* is obviously greater than $h(0)$, because $h(c)$ is a bell curve. This logic refers to the assertion that investors obviously invest in the two assets, including no-risk assets, i.e. money, at a constant rate through rebalancing. This assertion prevails when $h(0) = 0$, in which the expected value in the portfolio consisting of risky assets equals that of money. This theory says that adding partly money to a portfolio consisting of risky assets yields higher returns.

However, we need to specifically demonstrate how this logic works.

The system is described as

$$X_t = (1-c)\theta_t X_{t-1} + cX_{t-1},$$

$$I_t = (1-c)X_{t-1},$$

$$R_t = (1-c)\theta_t X_{t-1},$$

$$S_t = cX_{t-1} \qquad \text{and}$$

$$S_t + I_t = X_{t-1}.$$

Let us analyze the difference between the result of current investment and the investment in the subsequent period, which is expressed as

$$R_t - I_{t+1} = c(1-c)(\theta_t - 1)X_{t-1}, \qquad (14)$$

which shows that if $\theta_t > 1$, $I_{t+1} < R_t$, and that if $\theta_t < 1$, $I_{t+1} > R_t$.

We can economically interpret (14) this way: the investment in the subsequent period is smaller than the result in the current period, if the result of the current period investment

increases the value of the portfolio, and vice versa. This mechanism is deduced from $\theta_t \lesseqgtr 1$, that is, the fluctuation rates in the current price of risky assets. When the price is high, the subsequent investments decrease, and vice versa.
Additionally, this mechanism implies that investors utilize a price change rate in risky assets as an indicator in investment decisions by keeping c constant.

On the other hand, a change in the money value is described as

$$S_{t+1} - S_t = c(1-c)(\theta_t - 1)X_{t-1}, \tag{15}$$

which shows that if the current price of risky assets goes up, the money goes up in the subsequent term, and vice versa. This mechanism is automatic, based on the rational characteristics of investors who adopt a rebalancing strategy, and due to the existence of $0 < c < 1$.
In brief, when the current risky asset prices go up, in the subsequent period the investment in the risky assets decreases, while the money increases, and vice versa.

Based on (14) and (15), we can obtain

$$(S_t - S_{t+1}) + (R_t - I_{t+1}) = 0.$$

The above relationship implies that when investments decrease, which is deduced from the increase of risky assets prices, money correspondingly increases, and vice versa.

7. Concluding remarks

We have investigated rebalancing in terms of a portfolio consisting of risky assets and money. Moreover, we assume that the risky assets face two opposite tendencies, such as going up or going down in price, based on certain probabilities. In this situation, we deduce a theory, namely that in order to take advantage, investors maintain a constant proportion rate allocated between risky assets and money during every investment period. To maintain this constant means that investors rebalance the proportion according to fluctuations in prices. However, the optimal allocation rate is related to the combination among the two price rates and the frequency rate of the events. Furthermore, for the existence of an optimal rate the regions of the fluctuations in the prices are limited. If this rule is violated, the rebalancing theory becomes meaningless. The designers of a portfolio should be sensitive to this. Merton (1969) discusses the Wiener process, which relies on a normal distribution, while we deal with the Bernoulli trials, assuming a large number of trials, which does not depend on a normal distribution. However, in the case of Merton (1969) the explanations about the price mechanisms which rebalancing entails are not as clear as those in our discussions.

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