Trade warfare revisited: Trade and industrial policies when exporting and non-exporting firms co-exist

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Trade warfare revisited: Trade and industrial policies when exporting and non-exporting firms co-exist

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Abstract

This paper establishes a two-country, oligopoly trade model in which exporting firms and less-efficient non-exporting firms exist in each country, and firms have quadratic costs. Governments in the two countries conduct policy games involving various combinations of policy tools, including tariffs, production subsidies and export tax/subsidies. We show that the optimal policies have single-peaked relationships with the productivity differences between non-exporting and exporting firms. For example, because tariff can have either a U-shaped or an inverted U-shaped relationship with the productivity difference between firms depending upon other policy instruments available to the governments, improving the efficiency of the non-exporting firm can inhibit trade liberalisation. We also show that if each country can choose a discriminatory production subsidy policy, no trade occurs. Thus, combined trade and industrial policies may induce the decoupling of economies.

Key words: Trade and industrial policies; Non-exporting firm; Quadratic cost; Decoupling

JEL classification: F12; F13; O25; D43

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1 Introduction

China's industrial subsidies for its strategic and heavyweight sectors (Capital Trade Incorporated, 2009) were one cause of the US–China trade war initiated by the first Trump administration. These decoupling efforts were followed, under the Biden administration, by an industrial policy intended to encourage US domestic production of critical products previously imported from China on the basis of economic and national security (Foreign Policy (January 11, 2022)).

This paper investigates the consequences of the use of trade and industrial policies in a two-country oligopoly model in the presence of an inefficient domestic non-exporting firm. One of the key messages of the paper is that if two major countries use trade policies as industrial policies, they will ultimately block all trade with each other. This message not only reminds us of the recent "US–China trade war," but also forecasts the ultimate decoupling of the economies.

After the first Trump administration in the US raised tariffs in 2018 and the main trading partners of the US retaliated, many observers emphasised that these events were indicative of a revival of protectionism (Fajgelbaum et al., 2020). This revival has not been limited only to tariffs–the use of subsidies has also become more frequent. In reality, major countries often employ subsidies beyond those applying to research and development. For example, China grants production and export subsidies to its electronic vehicles and ship building industries and also uses them to encourage shale gas development (Bai et al., 2020; Barwick et al., 2019; Kalouptsidi, 2018; Shao et al., 2021; Shiming et al., 2020). In addition, observations of the US agricultural sector make it evident that it is protected by the US government, with farmers receiving generous subsidies (Kirwan, 2009).[1] Thus, it is apparent that protectionism is alive and flourishing.

[1]Governments not only provide export subsidies but also impose export taxes as part of their protectionist policies. For example, Pakistan imposed a tax on exports of raw cotton and yarn from 1988 to 1995 (Hudson and Ethridge, 1999). More recently, although for a shorter period of time, Russia imposed a tax on exports of wheat from 2007 to 2008 (Pall et al., 2013).

Theoretical studies on trade and industrial policies including subsidies are generally referred to as strategic trade theory studies. This literature has been expanding since the 1980s.[2] However, in the literature, two significant matters tend to be overlooked. Most studies (i) assume that firms have constant marginal costs; and (ii) do not consider the existence of non-exporting firms within an industry.

These two assumptions (i) and (ii) are central to the strategic trade theory models. First, Basu and Fernald (1987) empirically show that firms do not necessarily have constant marginal costs, but can have increasing marginal costs. If the firm's marginal cost is increasing, because an interaction among firms newly occurs, the nature of the optimal policies will differ from those when marginal cost is constant. Second, Bernard et al. (2007) and Freund and Pierola (2015) show that within many manufacturing industries, there are some firms that export but others that only supply products domestically. Furthermore, Clerides et al. (1998) empirically demonstrate that exporting firms are more productive than non-exporting firms. When we consider these factors, trade and industrial policy changes will lead to production reallocation not only among exporting firms but also between efficient and less-efficient firms. Therefore, the standard arguments regarding the optimal policies may alter.

We develop a two-country oligopoly trade model in which each country has three policy instruments – a tariff, a production tax/subsidy and an export tax/subsidy. In our model, firms have increasing marginal costs (that is, quadratic production costs) and efficient exporting firms

[2] As the literature on strategic trade theory in oligopoly markets is vast, we mention representative works only in this review. Brander and Spencer (1984) and Dixit (1984) conduct pioneering studies on tariff and subsidy policies. Brander and Spencer (1985) and Eaton and Grossman (1986) examine an export tax/subsidy in the export rivalry model. Krishna and Thursby (1991) consider export, production and consumption taxes/subsidies. Collie (1991) considers countervailing duties. Since the mid-1990s, the literature has expanded to consider the upstream market. Ishikawa and Spencer (1999) examine an export tax/subsidy in a two-country model of vertically related markets. Chang and Sugeta (2004) focus on vertical bargaining. Mizuno and Takauchi (2020) consider upstream price competition.

co-exist with less-efficient non-exporting firms in each country. Initially, each country determines its policies to maximise national welfare, then each firm subsequently decides on its output.

First, we find that when each country uses all three policy instruments, a tariff, an export tax and a production subsidy are optimal. To protect its domestic industry, each country imposes a tariff. To mitigate underproduction, each country provides a production subsidy. Promoting exports through the export subsidy increases the marginal cost of the efficient exporting firm, which decreases its domestic supply, and thus increases production by the less-efficient non-exporting firm. Hence, the export tax becomes optimal. This result occurs because of two effects: (a) the increase in marginal cost, and (b) the redistribution of production from the efficient to the less-efficient firm. We also show that all optimal policies can have an inverted U-shaped or a U-shaped relationship with respect to the parameter of productivity differences between the efficient and less-efficient firms. This indicates that a productivity parameter of a certain size maximises the optimal tariff and, thus, it implies that the productivity difference between non-exporting and exporting firms can inhibit trade liberalisation. This result, which arises from our two central assumptions, contributes to the trade policy literature. Table 1 illustrates the signs of the optimal policies in each scenario.

Table 1: The signs of the optimal policies in each case

	1	2	3	4	5	6	7
Tariff t_i	+	+	+	NA	NA	+	NA
Export subsidy e_i	−	NA	−	−	NA	NA	+/−
Production subsidy s_i	+	+	NA	+	+	NA	NA

Note: [i] "1" denotes the case in which all policies are used; "2" denotes the case in which a tariff and a production subsidy are used. [ii] $+(-)$ indicates that the sign is positive (negative); [iii] NA means not available.

Second, if the country commits to a tariff-only policy, this can result in a higher tariff rate compared with the case in which the country adopts a policy that combines a tariff with a production subsidy. This is because the tariff protects all domestic firms and, therefore, promotes domestic production. When the country combines the tariff with a production subsidy, it is not necessary to impose such a high tariff rate as under the tariff-only policy because the production subsidy promotes production by the domestic firms. Trade-distorting production subsidies are prohibited under the General Agreement on Tariffs and Trade (GATT)/World Trade Organization (WTO). Ironically, however, the policy combination involving a tariff and a production subsidy can result in lower trade barriers (i.e., a lower tariff rate) compared with the case of a tariff as the sole policy instrument. This result provides a new insight into trade policy.

Third, we present a unique result that is closely related to a potential real-world scenario. If each country can select discriminatory subsidies, we find that the trade volume between the two countries becomes zero and the optimal tariff and export subsidy also become zero. Each country can utilise four policies, which enables them to control production. As a result, the domestic price becomes equivalent to the marginal production cost and to the export price. For this reason, the trade volume, the tariff rate and the export subsidy are zero. This result has significant implications – that is, *it is possible that trade will be eliminated if the countries utilise combined trade and industrial policies*. This scenario is not necessarily conceptual only. In June 2018, the US imposed tariffs on Chinese products based on Section 301 of the Trade Act of 1974. In response, China imposed tariffs on US goods. Starting with these actions, both countries escalated the retaliatory policies, imposing non-tariff barriers and other trade restrictions in the situation referred to as the "US–China trade war" (Bown, 2021; Fajgelbaum et al., 2020). Our third result is reminiscent of the US–China trade war. The types of trade and industrial policies adopted depend strongly on the political regime. Therefore, our result has non-negligible implications for the real world.

This paper is related to two strands of literature: the literature that considers multiple trade and industrial policies (Cheng, 1988), and the literature that considers trade policies and firms' increasing marginal costs (Egli and Westermann, 2004; Long and Soubeyran, 1999).[3] Cheng (1988) examines the effects of a production tax/subsidy for the home firm and a tariff for the foreign exporting firm under the home country's general demand function. In a one-way trade model, he focuses on the relationship between the optimal policies and the degree of product differentiation between the home and foreign products. In contrast, we consider a two-country policy competition including an export tax/subsidy. Thus, our study differs substantially from that of Cheng (1988).

Egli and Westermann (2004) present a model in which one home firm and n foreign firms exist. All firms have a symmetric quadratic cost function. The home country gives a subsidy to the home firm and imposes a tariff on the foreign firms. The authors show that the optimal tariff has an inverted U-shape with respect to the number of foreign firms. However, because their model is substantially a one-way trade model, it differs from our model of a two-country policy competition. Long and Soubeyran (1999) emphasise cost heterogeneity in a quadratic cost function. They first consider the role of a firm-specific export tax when many firms exist in the third market. Furthermore, in the case of one-way trade when only foreign firms export, they consider a tariff and production subsidy. They mainly show that efficient foreign firms face a higher tariff. However, because they do not consider a two-country policy competition or the effects of the combined use of trade and industrial policies, their study is significantly different to ours.

The remainder of this paper proceeds as follows. Section 2 establishes the model. Section 3 discusses the three alternative policy scenarios, in which each country implements (1) all three policies, (2) policy combinations involving two policies and (3) a single policy. Section 4

[3]Several studies related to trade policy employ quadratic costs. Choi and Lim (2023) and Mukherjee and Sinha (2024) consider a tariff policy and an export tax/subsidy, respectively, in the context of quadratic costs. Although Mizuno and Takauchi (2020) analyse an export tax/subsidy, they assume that upstream firms have quadratic costs.

presents the rankings of each policy. In Section 5, we conduct a welfare analysis. In Section 6, we provide a discussion and extensions of the baseline model, with Sections 6.1 and 6.2 examining a many-firm case and discriminatory subsidies, respectively. Section 7 concludes the paper.

2 Model

We consider the two-way trade model of Brander and Krugman (1983). There are two countries, A and B, and each country i ($i = A, B$) has two firms that produce homogenous goods. In each country, there are two types of firms: one, which engages only in supplying the domestic market, is referred to as a *non-exporting firm*; the other, which supplies the domestic market and exports goods, is referred to as an *exporting firm*. To enter the foreign market, an entrant needs to possess the relevant knowledge and the means to distribute its product within the foreign market. Whereas the exporting firm has such knowledge and means, the non-exporting firm does not. Hence, the exporting firm supplies the domestic market and exports its products, whereas the non-exporting firm supplies its product exclusively to the local market.

We consider a linear quadratic cost function of firms, that is, $c_h q^h + \gamma_h (q^h)^2$ for $h = d, e$, where "d" denotes the "*non-exporting*" firm and "e" denotes the "*exporting*" firm. Here, $c_h \geq 0$ is a cost parameter of h firm that corresponds to its marginal cost when $\gamma_h = 0$. $\gamma_h > 0$ denotes the degree of production inefficiency and we assume that $\gamma_d > \gamma_e$. Hence, the non-exporting firm is less efficient than the exporting firm. q^h is the quantity of h firm.

The inverse demand function in country i ($i = A, B$) is given by

$$p_i = a - D_i = a - (q_i^d + q_{ii}^e + q_{ji}^e),$$

where p_i is the price of homogenous good, and D_i is the total sales in country i: that is, $D_i \equiv q_i^d + q_{ii}^e + q_{ji}^e$. Hence, the consumer surplus in country i is given by $CS_i = D_i^2/2$. Note that q_{ji}^e are the exports of country j's exporting firm ($j \neq i$).

Each country i ($i = A, B$) imposes a specific tariff, t_i, on imports. Hence, the profits of the d firm, π_i^d, and the e firm, π_i^e, are given by

$$\pi_i^d \equiv (p_i + s_i)q_i^d - c_d q_i^d - \frac{\gamma_d}{2}\big(q_i^d\big)^2,$$
$$\pi_i^e \equiv (p_i + s_i)q_{ii}^e + (p_j - t_j + s_i + e_i)q_{ij}^e - c_e(q_{ii}^e + q_{ij}^e) - \frac{\gamma_e\big(q_{ii}^e + q_{ij}^e\big)^2}{2} \quad \text{for} \quad i \neq j,$$

where s_i is a per unit production subsidy (if $s_i < 0$, it denotes a production tax), e_i is a per unit export subsidy (if $e_i < 0$, it is an export tax) and t_j is a specific import tariff of country j (if $t_j < 0$, it is an import subsidy of country j, where $j \neq i$ and $i, j = A, B$).

To focus on the effect that heterogeneity in the convexity of production cost has on the optimal policies, we assume a quadratic cost function here – that is, in the equations above regarding the profits of firms, we assume that $c_d = c_e = 0$, $\gamma_d = \gamma > 1$ and $\gamma_e = 1$. Hence, the cost function of non-exporting firm i is $\gamma(q_i^d)^2/2$ for $\gamma > 1$, and its output is q_i^d. The cost function of exporting firm i is $(q_{ii}^e + q_{ij}^e)^2/2$,[4] q_{ii}^e denotes the domestic supply, and q_{ij}^e denotes exports ($i, j = A, B$ and $i \neq j$). The quadratic cost function is a common setting that is frequently employed in oligopoly models (see, for example, Choi and Lim, 2023; Dastidar, 1995; Delbono and Lambertini, 2016; Mizuno and Takauchi, 2022, 2024; Mukherjee, 2014; Mukherjee and Sinha, 2024; Takauchi and Mizuno, 2022; Takauchi et al., 2024; von Weizsäcker, 1980).[5]

Now, we analyse a simple two-stage game. In the first stage of the game, each country i independently and simultaneously decides on the levels of its policy instruments to maximise its total surplus. In the second stage, non-exporting and exporting firms compete in a Cournot fashion in each market. The equilibrium concept is a subgame perfect Nash equilibrium; hence,

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[4]Even if we assume that the coefficient of the exporting firm's cost function is γ_e and $\gamma > \gamma_e > 0$, our main results do not alter.

[5]In reality, a decreasing-returns-to-scale technology is often observed in manufacturing sectors. Based on aggregate data of 34 manufacturing industries in the US, Basu and Fernald (1997) empirically demonstrate that a typical industry has decreasing returns to scale. In addition, Xu et al. (1994) find that decreasing returns to scale are observed in the transportation sector.

the game is solved using backward induction.

3 Trade and Industrial Policies

In this section, we consider the case in which countries have multiple policy instruments.

3.1 Tariff t_i, Export Subsidy e_i and Production Subsidy s_i

We first consider the case in which country i $(i = A, B)$ can utilise all three policy instruments, namely a tariff t_i, an export subsidy e_i and a production subsidy s_i.

Let $\mathbf{t} = (t_A, t_B,)$, $\mathbf{e} = (e_A, e_B)$ and $\mathbf{s} = (s_A, s_B)$. By inserting the second-stage outcomes into the total surplus of country i, and noting that the aggregate output in country i is $Q_i \equiv q_i^d + q_{ii}^e + q_{ij}^e$, we obtain

$$\begin{aligned} SW_i(\mathbf{t}, \mathbf{e}, \mathbf{s}) = CS_i(\mathbf{t}, \mathbf{e}, \mathbf{s}) + \pi_i^d(\mathbf{t}, \mathbf{e}, \mathbf{s}) + \pi_i^e(\mathbf{t}, \mathbf{e}, \mathbf{s}) \\ + t_i q_{ji}^e(\mathbf{t}, \mathbf{e}, \mathbf{s}) - e_i q_{ij}^e(\mathbf{t}, \mathbf{e}, \mathbf{s}) - s_i Q_i(\mathbf{t}, \mathbf{e}, \mathbf{s}), \end{aligned} \tag{1}$$

where $i \neq j$ and $i, j = A, B$.

From (1), by solving the welfare-maximisation problem of country i $\max_{t_i, e_i, s_i} SW_i(\mathbf{t}, \mathbf{e}, \mathbf{s})$,[6] we obtain the following optimal policies:

$$\begin{aligned} t_i^* &= \frac{a(\gamma - 1)(4\gamma + 7)}{10\gamma^3 + 82\gamma^2 + 159\gamma + 85} > 0, \\ e_i^* &= -\frac{a(5\gamma^3 + 21\gamma^2 + 45\gamma + 41)}{10\gamma^3 + 82\gamma^2 + 159\gamma + 85} < 0; \quad s_i^* = \frac{a(5\gamma^3 + 18\gamma^2 + 44\gamma + 45)}{10\gamma^3 + 82\gamma^2 + 159\gamma + 85} > 0, \end{aligned} \tag{2}$$

where "$*$" denotes the equilibrium value in the case where all three policies are implemented.

The equation (2) yields Proposition 1.

Proposition 1. *Suppose that each country i $(i = A, B)$ can implement tariff t_i, export subsidy e_i and production subsidy s_i. Then, (i) a tariff such that $t_i^* > 0$ is optimal; (ii) an export tax such that $e_i^* < 0$ is optimal; and (iii) a production subsidy such that $s_i^* > 0$ is optimal.*

[6] The second-order sufficient condition for welfare maximisation is always satisfied. See the Online Appendix.

The logic behind parts (i) and (iii) of Proposition 1 are intuitive. To protect the efficient domestic firm (i.e., the exporting firm), the country imposes a positive tariff $t_i^* > 0$ (part (i)). The oligopolist produces, ignoring consumer demand and, hence, underproduction occurs. To moderate such underproduction, the country introduces a positive production subsidy $s_i^* > 0$ and attempts to promote domestic production (part (iii)).

Part (ii) states that the export tax, $e_i^* < 0$, is optimal. The intuition is as follows. Consider that country i gives an export subsidy, $e_i > 0$, to its exporting firm. This leads the exporting firm to increase its export volume. However, because the marginal cost of the exporting firm is increasing, an increase in its export volume leads to a sharp rise in its production cost. This sharp increase in production cost leads the exporting firm to decrease its domestic supply. Then, the output of the less-effective non-exporting firm increases owing to strategic substitutability in the domestic market. Hence, the export promotion policy promotes the production activity of the less-efficient non-exporting firm rather than that of the efficient exporting firm. This result is in sharp contrast to the result in the case where the marginal cost is constant. As Lahiri and Ono (1988) show, assisting the less-efficient firm to increase production reduces national welfare. Therefore, in equilibrium, the country i imposes an export tax.

From the perspective of "production relocation", we emphasise the following mechanism. In equilibrium, $s_i^* > 0$ and $t_i^* > 0$. Hence, in equilibrium, because these two policies commonly protect the efficient domestic (exporting) firm as well as the inefficient (non-exporting) firm, the policies relocate production to the less-efficient firm. Thus, by setting $e < 0$, the country has an incentive to encourage the efficient (exporting) firm to divide its production so that it supplies the domestic market as well as exporting and, hence, to reduce production by the less-efficient firm. Therefore, the country imposes an export tax to achieve this.

It should be noted that the positive tariffs also affect the export terms of trade. Because of increasing marginal costs, the expansion of domestic supply by the exporting firm raises its own marginal costs and thus reduces its export sales. This results in a rise in the export price. Furthermore, the reduction in export sales by the home exporting firm prevents the foreign tariffs from shifting the profits from the home exporting firm to the foreign rivals.

We establish Proposition 2 from equation (2).

Proposition 2. *(i) The optimal tariff is maximised at* $\gamma = \gamma_t^* \simeq 4.48848$*.*

(ii) The optimal export tax is minimised at $\gamma = \gamma_{ex}^* \simeq 2.13124$*.*

(iii) The optimal production subsidy is minimised at $\gamma = \gamma_s^* \simeq 2.46098$*.*

Proof. See the Appendix.

Proposition 2 indicates that the optimal policies have a marked characteristic: that is, all three policies have a single-peaked relationship with the cost parameter γ. For example, in response to a degree of productivity difference within an industry, Proposition 2 points out that countries should maximise (minimise) their policy level. The intuition of Part (i) is as follows. When γ is small, the difference in the productivities of the domestic firms is small. In addition, the difference in productivity between the domestic non-exporting firm and the foreign exporting firm is small. Then, if γ increases, because the productivity of the domestic firms worsens and outputs decrease, the country has an incentive to raise its tariff to protect the domestic firms. By contrast, when γ is large, the production efficiency of the domestic non-exporting firm is low, and its productivity is lower than that of the foreign exporting firm. Thus, if there is an increase in γ and the country raises its tariff to protect its less productive domestic non-exporting firm, its welfare can fall (Lahiri and Ono, 1988). In this case, the optimal tariff decreases as γ increases. Hence, t_i^* has an inverted U-shape relationship with γ.

Part (i) also says that the optimal tariff can be affected by the differences in production efficiency within an industry. In reality, non-exporting and exporting firms co-exist within an industry, and their productivity differs. Regarding this, for example, we may consider that it is possible for a country to be a maximiser of the optimal tariff rate and, hence, the country may proceed with trade liberalisation if the difference in the production efficiency between the exporting and non-exporting firms is sufficiently large (or small).

Part (ii) is explained as follows. When γ is small, the productivity difference between domestic firms is small. Then, if the country raises the export tax in response to an increase in γ, the exporting firm decreases its exports and increases its domestic supply. This action has two negative effects. One is that it decreases the non-exporting firm's output because of the strategic substitutes relationship. This can lead to a welfare loss. The other negative effect is that it increases the marginal cost of exporting firm. The exporting firm compensates for a decrease in exports by increasing its domestic supply. However, the domestic supply is not affected by barriers, in contrast with exporting. Because the degree of the increase in the domestic supply is greater than the degree of the decrease in exports, the exporting firm's marginal cost rises. This can also lead to a welfare loss. Hence, the country lowers its export tax. When γ is large, the non-exporting firm is less efficient. If the country decreases the output of the non-exporting firm, because its national welfare can increase, it raises the export tax.

The intuition of part (iii) is as follows. The production subsidy has two opposing effects. One is a negative effect that increases firm's marginal cost arising from the promotion of production. The other is a positive effect that mitigates the underproduction of oligopolistic competition. Suppose that the country increases its production subsidy for an increase in γ when γ is small. Because no firm is particularly inferior to others in terms of productivity, the marginal cost of firms sharply increases owing to subsidies promoting production. Thus, the (negative) effect of an increase in marginal cost dominates the (positive) effect of the mitigation of underproduction (positive effect), and a welfare loss arises. Hence, the country lowers its production subsidy for an increase in γ. If γ is large, the non-exporting firm is less efficient. Then, because the (positive) effect of the mitigation of underproduction (positive effect) dominates the (negative) effect of the increase in marginal cost, the country raises its production subsidy for an increase in γ.

Constant marginal cost Here, we examine the case where firms have constant marginal costs. To better understand the problem, we establish the following simple condition:

$$c = c_d > c_e = 0 \quad \text{and} \quad \gamma_d = \gamma_e = 0. \quad \text{(CMC)}$$

Under the CMC and equation (1), we obtain the optimal policies t_i^L, e_i^L and s_i^L. Within the parameter range in which the welfare-maximisation problem has an economic meaning, the optimal policies have the following signs.[7]

$$t_i^L = \frac{3c}{8} > 0; \quad e_i^L = -\frac{(8a - 11c)}{16} < 0; \quad s_i^L = \frac{8a - 7c}{16} > 0.$$

Furthermore, $\partial t_i^L / \partial c > 0$, $\partial e_i^L / \partial c > 0$, and $\partial s_i^L / \partial c < 0$.

The signs of the optimal policies are the same as those in Proposition 1. However, the effect of the cost parameter in this case is in sharp contrast to the quadratic cost case (i.e., in the constant marginal cost case, the cost parameter is c). When marginal cost is constant, the optimal tariff rises as c increases. This is because a tariff reduction induces an increase of imports from the foreign firm and decreases the output of the non-exporting firm and the domestic supply of the exporting firm. Hence, the country raises its tariff in response to an increase in c. In this case, because the import tariff of the foreign country similarly rises, the country must promote exports through a decrease in the export tax. In addition, an increase in c implies that the non-exporting firm becomes less efficient, which reduces its incentives to promote production. The country decreases its production subsidy. As seen above, in the constant marginal cost case, the cost parameter c only has a monotonic effect on the optimal policies.

[7] To ensure a positive output, in this case, we require that $c/a < 8/15$.

3.2 Combination of Two Policy Instruments

The question now arises as to whether the character of policy alters when countries change to a combination that involves two policies To consider this matter, we focus on the following two-policy combinations: (a) a tariff and a production subsidy; (b) a tariff and an export subsidy; and (c) an export subsidy and a production subsidy.

(a) A tariff t_i and a production subsidy s_i: "ts". To obtain the equilibrium in Case (a), we substitute $\mathbf{e} = (0,0)$ into (1), and maximise it with respect to t_i and s_i. Then, we obtain the following optimal policies.

$$t_i^{ts} = \frac{a(16\gamma^3 + 81\gamma^2 + 139\gamma + 80)}{70\gamma^3 + 453\gamma^2 + 819\gamma + 446} > 0; \quad s_i^{ts} = \frac{a(13\gamma^3 + 43\gamma^2 + 122\gamma + 130)}{70\gamma^3 + 453\gamma^2 + 819\gamma + 446} > 0, \tag{3}$$

where "ts" denotes the equilibrium value when each country's policy is limited to the combination of a tariff and a production subsidy.

(b) A tariff t_i and an export subsidy e_i: "te". To obtain the equilibrium in Case (b), we substitute $\mathbf{s} = (0,0)$ into (1), and maximise it with respect to t_i and e_i. Then, we obtain the following optimal policies.

$$t_i^{te} = \frac{a(9\gamma^3 + 37\gamma^2 + 65\gamma + 45)}{(3\gamma + 5)(17\gamma^2 + 56\gamma + 45)} > 0; \quad e_i^{te} = -\frac{a\gamma(\gamma + 3)(3\gamma + 4)}{(3\gamma + 5)(17\gamma^2 + 56\gamma + 45)} < 0, \tag{4}$$

where "te" denotes the equilibrium value when each country's policy is limited to the combination of a tariff and an export subsidy.

(c) An export subsidy e_i and a production subsidy s_i: "es". To obtain the equilibrium in Case (c), we substitute $\mathbf{t} = (0,0)$ into (1), and maximise it with respect to e_i and s_i. Then, we obtain the following optimal policies.

$$e_i^{es} = -\frac{a(8\gamma^3 + 35\gamma^2 + 71\gamma + 59)}{16\gamma^3 + 130\gamma^2 + 243\gamma + 130} < 0; \quad s_i^{es} = \frac{a(8\gamma^3 + 31\gamma^2 + 73\gamma + 61)}{16\gamma^3 + 130\gamma^2 + 243\gamma + 130} > 0, \tag{5}$$

where "es" denotes the equilibrium value when each country's policy is limited to export and production subsidies.

Hence, from (3)–(5), we establish Proposition 3.

Proposition 3. *Suppose that each country i ($i = A, B$) can implement a policy combination involving two policy instruments. Then, (a) a tariff $t_i^{ts} > 0$ is optimal; and a production subsidy $s_i^{ts} > 0$ is optimal. (b) a tariff $t_i^{te} > 0$ is optimal; and an export tax $e_i^{te} < 0$ is optimal. (c) an export tax $e_i^{es} < 0$ is optimal; and a production subsidy $s_i^{es} > 0$ is optimal.*

The signs of the optimal policies are the same for the two-policy combination cases as those for the case where all three policies can be conducted. Then, the logic behind the export tax is also the same as in Proposition 1.

Case (a) has the same logic as the three-policy case. The country has an incentive to protect its domestic exporting firm by setting $t_i > 0$, and it has an incentive to resolve domestic underproduction through $s_i > 0$. In Case (b), the same logic operates and, therefore, the same result holds as in the case of three policies. In Case (c), the logic is also the same as in the case where three policies are conducted.

3.3 Single Policy Instrument

The optimal policy can have different characteristics depending on whether multiple policies are conducted or a single policy. Next, therefore, we consider the case in which each country commits to using a single policy instrument.

First, we consider a tariff. In (1), by setting $\mathbf{e} = \mathbf{s} = (0, 0)$, the optimal tariff policy of country i, t_i^o, becomes

$$t_i^O = \frac{a(33\gamma^3 + 133\gamma^2 + 212\gamma + 130)}{159\gamma^3 + 771\gamma^2 + 1232\gamma + 650} > 0. \tag{6}$$

Equation (6) yields the following result.

Proposition 4. *(i) Suppose that* $1 \leq \gamma \leq \gamma_t^O \simeq 1.40847$. *An increase in the production efficiency of the non-exporting firm, i.e., a decrease in* γ*, increases the tariff rate. (ii) Suppose that* $\gamma > \gamma_t^O$*. An increase in the production efficiency of the non-exporting firm lowers the tariff rate.*

Proof. Differentiating t_i^O with respect to γ, we obtain $\partial t_i^O/\partial\gamma = \frac{8a(537\gamma^4+1737\gamma^3+343\gamma^2-3445\gamma-2795)}{(159\gamma^3+771\gamma^2+1232\gamma+650)^2}$. By solving $\partial t_i^O/\partial\gamma \geq 0$ for γ, we find that $\partial t_i^O/\partial\gamma \geq (<) 0$ if $\gamma \geq (<) \gamma_t^O \simeq 1.40847$. $\square$

The logic behind Proposition 4 is as follows. An increase in γ decreases the output of the non-exporting firm and the consumer surplus. The reduction in the consumer surplus is partly compensated by an increase in the domestic supply of the exporting firm and the foreign firms' exports. Whether the country i increases or decreases its tariff depends on whether the country prefers to rely on an increase in the domestic supply of the exporting firm or on an increase in imports from the foreign firm.

When γ is small, because the efficiency of the non-exporting firm is high, its output decreases significantly as γ increases. Then, if country i compensates for the loss of consumer surplus using only an increase in the domestic supply of the exporting firm, the outcome is not desirable for welfare because the marginal cost of the exporting firm increases considerably. Then, lowering the tariff to promote imports from the foreign firm, rather than increasing the exporting firm's marginal cost, is optimal for welfare.

When γ is large, the consumer surplus does not decrease so significantly even if γ increases. Hence, even if the country i compensates for the loss of consumer surplus mainly by increasing the domestic supply of the exporting firm, the firm's marginal cost does not increase as much as in the case above. Thus, the country increases its tariff.

Next, we consider the export subsidy. By setting $\mathbf{t} = \mathbf{s} = (0, 0)$ in (1), the optimal export subsidy (or tax) of country i, e_i^o, becomes

$$e_i^O = -\frac{a(\gamma-2)(3\gamma^2+11\gamma+11)}{69\gamma^3+302\gamma^2+437\gamma+210}. \tag{7}$$

We immediately find the following result from (7).

Proposition 5. *If γ is smaller than 2, the optimal policy is an export subsidy $e_i^O > 0$; If γ is larger than 2, the optimal policy is an export tax $e_i^O < 0$.*

The intuition of Proposition 5 is simple. If γ is sufficiently small (i.e., $\gamma < 2$), the non-exporting firm is not relatively less efficient than the exporting firm. Then, in contrast with the case of Proposition 1, even if country i subsidises its exporting firm, indirectly promoting the production of the non-exporting firm, a welfare loss does not occur. That is, by promoting the exports of the exporting firm, the home country can take rent from the foreign country, so the export subsidy appears in equilibrium.

Finally, we consider a production subsidy. By setting $\mathbf{t} = \mathbf{e} = (0,0)$ in (1), the optimal production subsidy of country i, s_i^O, becomes

$$s_i^O = \frac{2a(12\gamma^3+52\gamma^2+128\gamma+115)}{96\gamma^3+593\gamma^2+1042\gamma+554} > 0. \tag{8}$$

From (8), we obtain the following remark.

Remark. *(i) Suppose that $1 \leq \gamma \leq \gamma_s^O \simeq 4.89589$. An increase in the production efficiency of the non-exporting firm, i.e., a decrease in γ, increases the production subsidy. (ii) Suppose that $\gamma > \gamma_s^O$. An increase in the production efficiency of the non-exporting firm decreases the production subsidy.*

Proof. First, $s_i^O\big|_{\gamma=1} = \frac{614a}{2285} \simeq 0.268709a$ and $s_i^O \to a/4$ as $\gamma \to \infty$. Second, differentiating s_i^O with respect to γ, we obtain $\partial s_i^O/\partial\gamma = \frac{12a(354\gamma^4+72\gamma^3-5816\gamma^2-13129\gamma-8153)}{(96\gamma^3+593\gamma^2+1042\gamma+554)^2}$. Solving $\partial s_i^O/\partial\gamma \leq 0$ for γ, we have $\gamma \leq \gamma_s^O \simeq 4.89589$. Thus, $\partial s_i^O/\partial\gamma \leq (>) 0$ if $\gamma \leq (>) \gamma_s^O$. □

Suppose that the non-exporting firm is not relatively less efficient, that is, $\gamma < \gamma_s^O$. A decrease in γ results in the marginal costs of firms falling if their outputs do not alter. To promote the production of firms, the country has an incentive to subsidise firms when γ decreases. Conversely, if $\gamma > \gamma_s^O$, the non-exporting firm is less efficient. In this case, subsidizing the firms implies that the country helps even up to the production in the less-efficient firm and, hence, it can decrease welfare. Hence, the country decreases the subsidy level for a decrease in γ.

4 Ranking of Each Policy

Now, we examine the ranking in the magnitude of each policy. We find that the ranking of the two subsidies does not change in relation to the size of γ. However, in the case of the tariff policy, the ranking does alter depending on the size of γ; this could have a significant policy implication.

From equations (2)–(4) and (6), we establish the following result.

Proposition 6. *If the production efficiency of the non-exporting firm is sufficiently high, the single-policy case involving the use of a tariff yields the highest tariff rate among all the cases. Otherwise, if the production efficiency of the non-exporting firm is not sufficiently high, the policy combination of a tariff and a production subsidy yields the highest tariff rate among all the cases. Formally,*

(i) $t_i^O \geq t_i^{ts} > t_i^{te} > t_i^*$ *for* $\gamma \leq \frac{1+4\sqrt{13}}{9} \simeq 1.71358$. *(ii)* $t_i^{ts} > t_i^O > t_i^{te} > t_i^*$ *for* $\gamma > \frac{1+4\sqrt{13}}{9}$.

Proof. See the Appendix.

[Figure 1 around here]

Proposition 6 has significant policy implications. One is that employing all three policies results in the lowest tariff rate (i.e., trade barrier) of all the policy cases. The other implication is that using a tariff-only policy achieves a trade barrier that is at the highest level among all the cases. This is because, although the trade-distorting production and export subsidies are prohibited within the GATT/WTO framework, the policy combination of these prohibitive subsidies and an import tariff can lower the tariff rate – that is, it can lower trade barriers.

We first compare t_i^O with t_i^{ts}. (See Figure 1.) Suppose that γ is small (i.e., $\gamma \leq \frac{1+4\sqrt{13}}{9}$). Then, in the case where country i uses a tariff-only policy t_i, the tariff can play a corrective role in preventing underproduction (i.e., it addresses the domestic oligopoly distortion). This is because the tariff protects all domestic firms and, hence, all domestic firms increase the domestic supply. Thus, by setting a higher tariff and partly shutting the foreign exporting firm out of the domestic market, country i attempts to increase the domestic supply of the domestic non-exporting and exporting firms. By contrast, in the case where country i can use a tariff in combination with a production subsidy s_i, because s_i increases the domestic supply of all domestic firms, a high tariff is not required. Hence, $t_i^O \geq t_i^{ts}$.

When γ is large (i.e., $\gamma > \frac{1+4\sqrt{13}}{9}$), because s_i is a uniform subsidy, it does not sufficiently increase the efficient exporting firm's domestic supply. As the tariff t_i increases the domestic exporting firm's local supply, it increases that firm's marginal cost. Thus, an increase in t_i reduces the exports of the domestic exporting firm because of the increase in its marginal cost. Hence, to decrease the exports of the domestic exporting firm, country i sets a higher tariff. As a result, $t_i^O < t_i^{ts}$ holds.

Next, let us consider other cases. In the other cases, the ranking does not depend on the size of γ. In equilibrium, e_i is an export tax. Thus, the export tax works to divert the efficient exporting firm's production to the domestic market. By diverting a certain portion of exports to domestic production, the exports of the foreign rival firm decrease. Hence, the country does not have an incentive to impose a high tariff. Consequently, $t_i^{te} < \min\left\{t_i^O, t_i^{ts}\right\}$ holds. If the country can employ both production and export subsidies, it can indirectly protect its domestic firms through these two subsidies. Hence, the country does not need to apply a high tariff and the tariff rate is the lowest among all the cases.

Regarding the export tax, we obtain the following ranking.

Lemma 1. $|e_i^{es}| > |e_i^*| > |e_i^{te}| > |e_i^O|$ *for all* $\gamma \in (1, \infty)$.

Lemma 1 can be explained as follows. When the country does not impose a tariff, it is immune from concerns about a welfare loss arising from a decrease in imports, and therefore it sets a higher export tax. Hence, $|e_i^{es}| > |e_i^*|$. If the country does not provide a production subsidy to its domestic firms, the marginal cost of those firms becomes high. Because the marginal cost of exporting rises greatly, the country cannot increase the export tax. Thus, $|e_i^*| > |e_i^{te}|$ holds. Through the protection of domestic firms, the tariff policy can promote the production of less-efficient firms. The production subsidy also promotes the production of the less-efficient firms because it lowers the marginal cost. If there are no tariff and production subsidy policies, the country has no motivation to set a higher export tax. This is because there is no need to stimulate the exporting firm (the efficient firm) to produce a larger domestic supply by raising the marginal cost of exports. Therefore, $|e_i^{te}| > |e_i^O|$. Lemma 1 is illustrated in Figure 2.

[Figure 2 around here]

The ranking of the production subsidy is as follows.

Lemma 2. $s_i^{es} > s_i^* > s_i^O > s_i^{ts}$ *for all* $\gamma \in (1, \infty)$.

If the country imposes a tariff, the imports fall. This induces an increase in the domestic supply of the exporting firm and the output of the non-exporting firm in that country and, hence, total outputs can increase. Because the country does not require a high production subsidy, $s_i^{es} > s_i^*$ holds. In equilibrium, the export tax is imposed and the country impedes exports. The marginal cost of exports increases, so the exporting firm decreases its exports and increases its domestic supply. This action inhibits the production of the less-efficient non-exporting firm. Hence, the country has an incentive to offer a larger production subsidy and, thus, $s_i^* > s_i^O$. If the country imposes only a tariff (i.e., a combination involving a tariff and a production subsidy), it protects its domestic industry and ultimately promotes the production of the less-efficient non-exporting firm. This reduces the incentive of the country to promote production, so $s_i^O > s_i^{ts}$ holds. These results are shown in Figure 3.

[Figure 3 around here]

5 Welfare Analysis

In this section, we compare total surpluses. The equilibrium total surplus under each regime is provided in the appendices. (See Appendix A.1.)

From (9) and (10) in Appendix A.1, we obtain Lemma 3.

Lemma 3. *(i)* **Multiple-policy case:** *If country i ($i = A, B$) can use multiple policy instruments, the welfare ranking is $SW_i^{es} > SW_i^* > SW_i^{ts} > SW_i^{te}$ for all $\gamma \in (1, \infty)$.*
(ii) **Single-policy case:** *If the country i can commit itself to the use of only a single policy instrument, the welfare ranking is $SW_i^s > SW_i^e > SW_i^t$ for all $\gamma \in (1, \infty)$.*

Let us consider Part (i). Suppose that the country does not impose a tariff only (i.e., it combines the tariff with other policies). In equilibrium, the export tax is imposed, which impedes export activity. However, the export tax raises the marginal cost of exporting and increases the exporting firm's domestic supply, so it reduces the output of the less-efficient (non-exporting) firm assuming it produces strategic substitutes. Furthermore, the production subsidy can resolve underproduction by the domestic oligopoly. The tariff policy protects all firms, including the uncompetitive, less-efficient (non-exporting) firm and thus it results in a welfare loss. Hence, $SW_i^{es} > SW_i^*$. If the country removes its export tax ($e_i^* < 0$), the marginal cost of exports declines and exports increase, which leads the exporting firm to decrease its domestic supply. Moreover, an increase in exports implies that imports from the foreign country rise. This decreases the outputs of the domestic firms and leads to a welfare loss. Thus, $SW_i^* > SW_i^{ts}$. If the country imposes a tariff, it decreases imports from the foreign country and increases domestic production. Although this promotion of domestic production includes the output of the less-efficient firm, the domestic underproduction ceases. By contrast, if the country combines an export tax with a tariff policy, the marginal cost of exports rises and the exporting firm increases its domestic supply, which results in the policy decreasing the output of the non-exporting firm. However, the increase in the domestic supply of the exporting firm and a decrease in the output of the non-exporting firm partially cancel each other out. As a result, $SW_i^{ts} > SW_i^{te}$.

Part (ii) is intuitive. First, because the production subsidy resolves underproduction caused by the domestic oligopoly, it maximises welfare. The export subsidy only promotes the export activities of the efficient exporting firm, and its welfare-improving effect is a halfway compared with the production subsidy. The tariff protects all firms including the less-efficient firm, which makes the welfare loss relatively large. Then, the ranking of this case is worse than the other cases.

From Lemmas 1 and 3, we establish Proposition 7

Proposition 7. *Suppose that the country cannot use the tariff policy. Then, if the country imposes a higher tax on the exports of the efficient exporting firm and the production efficiency of the non-exporting firm is sufficiently small, that country's welfare becomes higher. That is,* $SW_i^{es} > SW_i^s$ *if* $\gamma < \gamma_0 \simeq 1.45555$.

Proof. See the Appendix.

Suppose that γ is small. Although the production inefficiency of the non-exporting firm becomes small, it definitely exists. Then, if the government gives a production subsidy to the domestic firms, because this assists the non-exporting firm, a welfare loss still occurs. If the government provides an export subsidy under the two-policy combination scheme of export and production subsidies, the export subsidy becomes a tax in equilibrium, which yields tax revenue. This can lead to a welfare gain. Hence, $SW_i^{es} > SW_i^s$.

When γ is large, the production inefficiency of the non-exporting firm is large and, thus, it has a smaller market share. In this case, if the country charges an export tax, the marginal cost of exports rises and the exporting firm decreases its exports. This makes the exporting firm's domestic supply larger, so the output of the non-exporting firm decreases. However, the negative effects of an increase in the marginal cost of exports dominate the positive effects of a reduction in the output of the less-efficient firm because the output of less-efficient firm becomes very small. Therefore, $SW_i^{es} < SW_i^s$ holds. (See Figure 4.)

[Figure 4 around here]

6 Discussion and Extensions

In this section, we provide two distinctive arguments. One concerns the general situation in which $m(> 1)$ non-exporting firms and $n(> 1)$ exporting firms belong to each country. The other concerns the situation in which each country can use discriminative production subsidies.

6.1 Many Firms: m Non-Exporting and n Exporting Firms

Let $\alpha \equiv a - c_d > 0$ and $c \equiv c_d - c_e > 0$. By maximising each country's total surplus under the same conditions as in the previous section (with the exception of the number of non-exporting and exporting firms), we obtain the following.

$$e_i^*(m,n,\gamma) = -\frac{\alpha[n\gamma N + \widehat{m}(\gamma{+}1)(m{+}\widehat{N})]}{n\gamma^2 N + \widehat{m}[\widehat{m}(m+\widehat{N}) + \gamma(\gamma+1)NM]}$$
$$-\frac{cn[(\gamma{+}1)(n{+}\gamma{+}1{+}\widehat{m}(m{+}\widehat{N})) - (N{+}\gamma)(m{+}(\gamma{+}1)N{+}\widehat{N})]}{(\gamma+1)(n\gamma^2 N + \widehat{m}[\widehat{m}(m+\widehat{N}) + \gamma(\gamma+1)NM])} \gtreqless 0,$$
$$s_i^*(m,n,\gamma) = \frac{\alpha[n\gamma N + \widehat{m}(\gamma+1)(m+\widehat{N})] - cn[(m+\gamma)\,N + (\gamma+1)(m+\widehat{N})]}{n\gamma^2 N + \widehat{m}[\widehat{m}(m+\widehat{N}) + \gamma(\gamma+1)NM]} > 0,$$
$$t_i^*(m,n,\gamma) = \frac{cn(N-1)(\widehat{N} - m\gamma)}{(\gamma+1)(n\gamma^2 N + \widehat{m}[\widehat{m}(m+\widehat{N}) + \gamma(\gamma+1)NM])} > 0,$$

where $\widehat{m} \equiv m+n$, $M \equiv 2m+2n+2\gamma+1$, $N \equiv m+n+\gamma+1$, and $\widehat{N} \equiv (2\gamma+1)N$.

The logic behind a positive tariff $t_i^*(m,n,\gamma) > 0$ and production subsidy $s_i^*(m,n,\gamma) > 0$ is the same as in the previous section.[8]

In contrast with the previous section, it is possible that $e_i^*(m,n,\gamma)$ has a positive value – that is, $e_i^*(m,n,\gamma)$ can become an export subsidy.

As the simplest case, we can consider the following. Let $\gamma = 0$. Then,

$$e_i^*(m,n,\gamma=0) > 0 \quad \text{if} \quad n+1 > m.$$

The export promotion achieved by the export subsidy implies that imports increase, given the symmetry of both countries. However, because the number of exporting firms is relatively small, the reduction of domestic production caused by the increase in imports from the foreign country is small. Hence, the welfare-enhancing effect of the export promotion dominates the negative effect of the reduced domestic production. Thus, the optimal policy is an export subsidy. (For the case of $\gamma > 0$, see the Online Appendix.)

[8]To ensure positive production, it is necessary that $q_i^{d*}(m,n,\gamma) > 0$. By setting $q_i^{d*}(m,n,\gamma) > 0$, $t_i^*(m,n,\gamma) > 0$ and $s_i^*(m,n;\gamma) > 0$ hold.

6.2 Discriminatory Subsidies

For clarity, in this subsection, let $q_i^d \equiv z_i$, $q_{ii}^e \equiv y_i$ and $q_{ij}^e \equiv x_i$. The profits of the non-exporting and exporting firm, respectively, become:

$$\pi_i^d \equiv (p_i + s_{iz})z_i - \frac{\gamma}{2}z_i^2,$$
$$\pi_i^e \equiv (p_i + s_{iy})y_i + (p_j - t_j + s_{ix})x_i - \frac{(y_i + x_i)^2}{2} \quad \text{for} \quad i \neq j.$$

Here, s_{iz} denotes the production subsidy on the non-exporting firm's output. s_{iy} is the production subsidy on the exporting firm's domestic supply. s_{ix} is the export subsidy on the exporting firm's exports. Everything else remains the same as in the previous section.

In this scenario, we obtain the following results.

Proposition 8. *Suppose that country i can implement discriminatory subsidy policies. That is, it applies subsidy s_{iz} to the domestic non-exporting firm's production, subsidy s_{iy} to the domestic exporting firm's local supply and subsidy s_{ix} to the domestic exporting firm's exports. Then, (i) the export volume is zero but other production is positive and (ii) the optimal tariff and the optimal export subsidy become zero, but the other optimal subsidies are positive. Formally,*

$$\textit{(i)}\ x_i = 0 \quad \textit{and} \quad y_i = \frac{a\gamma}{2\gamma+1} > \frac{a}{2\gamma+1} = z_i,$$
$$\textit{(ii)}\ t_i = 0 = s_{ix} \quad \textit{and} \quad s_{iy} = \frac{a\gamma}{2\gamma+1} > \frac{a}{2\gamma+1} = s_{iz}.$$

Proof. See the Online Appendix.

The logic behind Proposition 8 is intuitive. When the four policies are implemented, each country treats its two domestic firms as if they were a single firm. That is, the country can perfectly control domestic production. Then, the domestic price equals the marginal production cost. Now, given that the two countries are symmetrical, the domestic price also equals the export price. As a result, trade volume becomes zero, and the tariff rate is zero. Each country only produces for its domestic market.

7 Conclusion

This paper has examined trade and industrial policy competition using a two-country trade model with efficient exporting and less-efficient non-exporting firms. Both types of firms have increasing marginal costs (quadratic production costs). Each country implements tariffs, export tax/subsidies and production tax/subsidies. Our main findings are summarised as follows. When each country decides on three policies, the optimal policies have a single-peaked relationship (i.e., a U- or inverted U-shaped curve) with the productivity difference between the non-exporting firm and the exporting firm. For example, in the case where the relationship of the tariff and the productivity different is expressed by an inverted U-shaped curve, an improvement of the non-exporting firm's productivity can impede trade liberalisation. Surprisingly, if each country implements a tariff-only policy, it results in a higher tariff level compared with the situation where each country uses a combination of a production subsidy and a tariff. As trade-distorting production subsidies are prohibited by GATT/WTO, it may be said that this result is somewhat ironic. The last result that we mentioned, involving the four policies, is the most realistic. This is because, if each country can differentiate its production subsidies, no trade occurs between the two countries. This implies that multiple trade and industrial policies can cause decoupling of economies. The US–China trade war is a prime example of such a decoupling and, hence, our model may even forecast the future of the world economy.

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Appendix

A.1. Welfare

When the country i ($i = A, B$) can conduct multiple policies, its national welfare is given by

$$\begin{aligned} SW_i^* &= \frac{a^2(\gamma+1)(50\gamma^5+770\gamma^4+4412\gamma^3+11603\gamma^2+14212\gamma+6585)}{2(10\gamma^3+82\gamma^2+159\gamma+85)^2}, \\ SW_i^{es} &= \frac{a^2(128\gamma^6+2080\gamma^5+12970\gamma^4+39066\gamma^3+61274\gamma^2+48569\gamma+15487)}{2(16\gamma^3+130\gamma^2+243\gamma+130)^2}, \\ SW_i^{te} &= \frac{a^2(11\gamma+18)(12\gamma^3+67\gamma^2+133\gamma+90)}{2(17\gamma^2+56\gamma+45)^2}, \\ SW_i^{ts} &= \frac{a^2(2400\gamma^6+31831\gamma^5+170493\gamma^4+471009\gamma^3+708575\gamma^2+551800\gamma+174324)}{2(70\gamma^3+453\gamma^2+819\gamma+446)^2}. \end{aligned} \tag{9}$$

When country i commits itself to using a single policy instrument, its national welfare is given by

$$\begin{aligned} SW_i^s &= \frac{a^2(24\gamma^2+101\gamma+98)(192\gamma^4+1636\gamma^3+4881\gamma^2+6081\gamma+2698)}{2(96\gamma^3+593\gamma^2+1042\gamma+554)^2}, \\ SW_i^e &= \frac{a^2(42\gamma^2+115\gamma+76)(54\gamma^4+356\gamma^3+915\gamma^2+1061\gamma+460)}{2(69\gamma^3+302\gamma^2+437\gamma+210)^2}, \\ SW_i^t &= \frac{2a^2(3\gamma+5)(17\gamma+26)(57\gamma^4+406\gamma^3+1118\gamma^2+1389\gamma+650)}{(159\gamma^3+771\gamma^2+1232\gamma+650)^2}. \end{aligned} \tag{10}$$

A.2. Proofs of Propositions

Proof of Proposition 2. First, $t_i^* = 0$ for $\gamma = 1$ and $t_i^* \to 0$ as $\gamma \to \infty$. Differentiating t_i^* with respect to γ, we obtain $\partial t_i^*/\partial\gamma = -\frac{4a(10\gamma^4+15\gamma^3-150\gamma^2-457\gamma-342)}{(10\gamma^3+82\gamma^2+159\gamma+85)^2}$. Solving $\partial t_i^*/\partial\gamma \leq 0$ for γ, we find that $\partial t_i^*/\partial\gamma \geq (<)\ 0$ if $\gamma \leq (>)\ \gamma_t^* \simeq 4.48848$.

Second, $\lim_{\gamma\to\infty} e_i^* = -a/2 < -a/3 = e_i^*\big|_{\gamma=1}$. Differentiating e_i^* with respect to γ, we obtain $\partial e_i^*/\partial\gamma = -\frac{2a(100\gamma^4+345\gamma^3-153\gamma^2-1577\gamma-1347)}{(10\gamma^3+82\gamma^2+159\gamma+85)^2}$. Solving the inequality $\partial e_i^*/\partial\gamma \geq 0$ for γ, we obtain $\partial e_i^*/\partial\gamma \geq (<)\ 0$ if $\gamma \leq (>)\ \gamma_{ex}^* \simeq 2.13124$.

Finally, $\lim_{\gamma\to\infty} s_i^* = a/2 > a/3 = s_i^*\big|_{\gamma=1}$. Differentiating s_i^* with respect to γ, we obtain $\partial s_i^*/\partial\gamma = \frac{a(230\gamma^4+710\gamma^3-821\gamma^2-4320\gamma-3415)}{(10\gamma^3+82\gamma^2+159\gamma+85)^2}$. Solving the inequality $\partial s_i^*/\partial\gamma \geq 0$ for γ, we have $\partial s_i^*/\partial\gamma \geq (<)\ 0$ if $\gamma \geq (<)\ \gamma_s^* \simeq 2.46098$. These results imply Proposition 2. □

Proof of Proposition 6. First, $\bar{t}_i$ and $\tilde{t}_i$ yield

$$\bar{t}_i - \tilde{t}_i = \frac{2a(93\gamma^6 + 756\gamma^5 + 2770\gamma^4 + 4775\gamma^3 + 3014\gamma^2 - 685\gamma - 1035)}{(3\gamma+5)(17\gamma^2+56\gamma+45)(70\gamma^3+453\gamma^2+819\gamma+446)} > 0.$$

Second, $\tilde{t}_i > t_i^*$, that is

$$\tilde{t}_i - t_i^* = \frac{2a(5\gamma^3 + 18\gamma^2 + 44\gamma + 45)(9\gamma^3 + 58\gamma^2 + 107\gamma + 60)}{(3\gamma+5)(17\gamma^2+56\gamma+45)(10\gamma^3+82\gamma^2+159\gamma+85)} > 0.$$

Third, from t_i^o and $\tilde{t}_i$, we obtain

$$t_i^o - \tilde{t}_i = \frac{2a\gamma(\gamma+3)(2\gamma+3)(3\gamma+4)(21\gamma^2+70\gamma+55)}{(3\gamma+5)(17\gamma^2+56\gamma+45)(159\gamma^3+771\gamma^2+1232\gamma+650)} > 0.$$

Comparing t_i^o and $\bar{t}_i$, we obtain

$$t_i^o - \bar{t}_i = -\frac{2a(\gamma+1)(9\gamma^2-2\gamma-23)(13\gamma^3+43\gamma^2+122\gamma+130)}{(70\gamma^3+453\gamma^2+819\gamma+446)(159\gamma^3+771\gamma^2+1232\gamma+650)}.$$

Thus, $\text{sign}\{t_i^o - \bar{t}_i\} = \text{sign}\{23 + 2\gamma - 9\gamma^2\}$. From this, we have $23 + 2\gamma - 9\gamma^2 \geq(<) 0$ if $\gamma \leq(>) (4\sqrt{13}+1)/9$. Hence, Proposition 6 holds. □

Proof of Proposition 7. First, $SW_i^{es} - SW_i^{ts} > 0$ for all $\gamma \in (1,\infty)$; $SW_i^s - SW_i^{ts} > 0$ for all $\gamma \in (1,\infty)$. Second, from $SW_i^s - SW_i^{es}$, we have

$$SW_i^s - SW_i^{es} = \frac{a^2(8\gamma^3+35\gamma^2+71\gamma+59)\begin{bmatrix} 27648\gamma^8+369720\gamma^7+1835577\gamma^6+3989086\gamma^5 \\ +2318860\gamma^4 - 5860550\gamma^3 - 12275762\gamma^2 \\ -9004976\gamma - 2413394 \end{bmatrix}}{(16\gamma^3+130\gamma^2+243\gamma+130)^2(96\gamma^3+593\gamma^2+1042\gamma+554)^2}.$$

By numerical calculation, $SW_i^s - SW_i^{es} \leq(>) 0$ for $\gamma \leq(>) \gamma_0 \simeq 1.45555$. From Lemma 2, e_i^{es} is the highest export tax. These results imply Proposition 7. □

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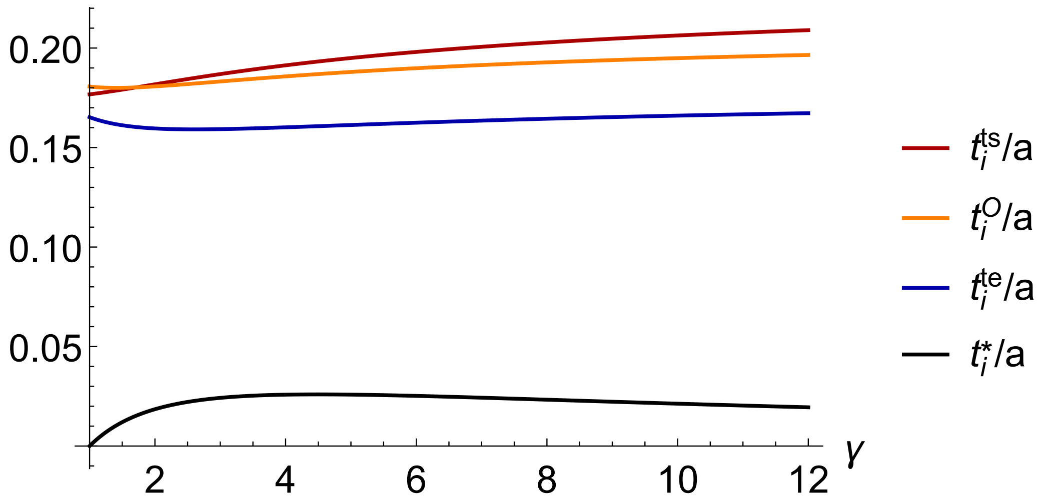

Figure 1: All tariffs

Note: Vertical axis is tariff rate

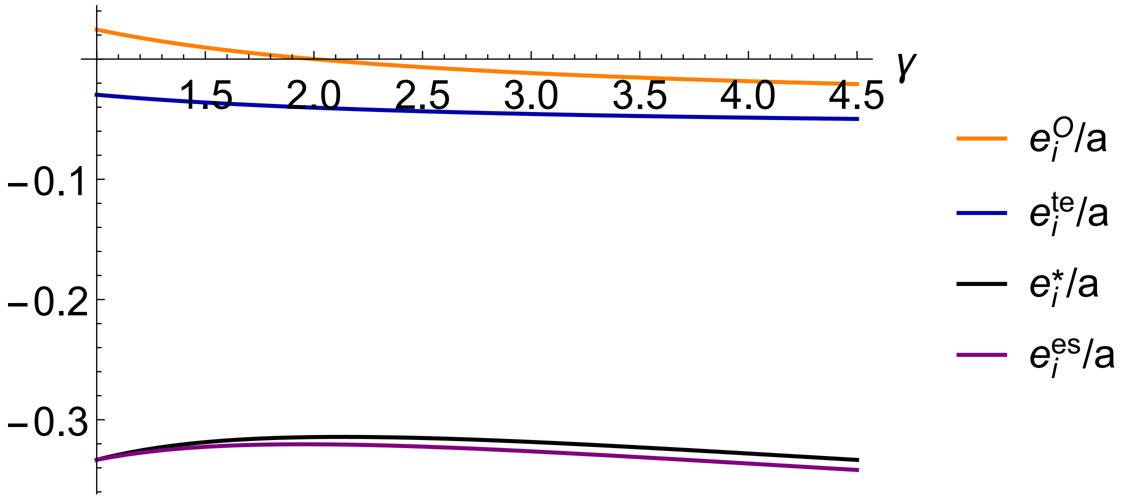

Figure 2: All export subsidy/tax

Note: Vertical axis is tax rate

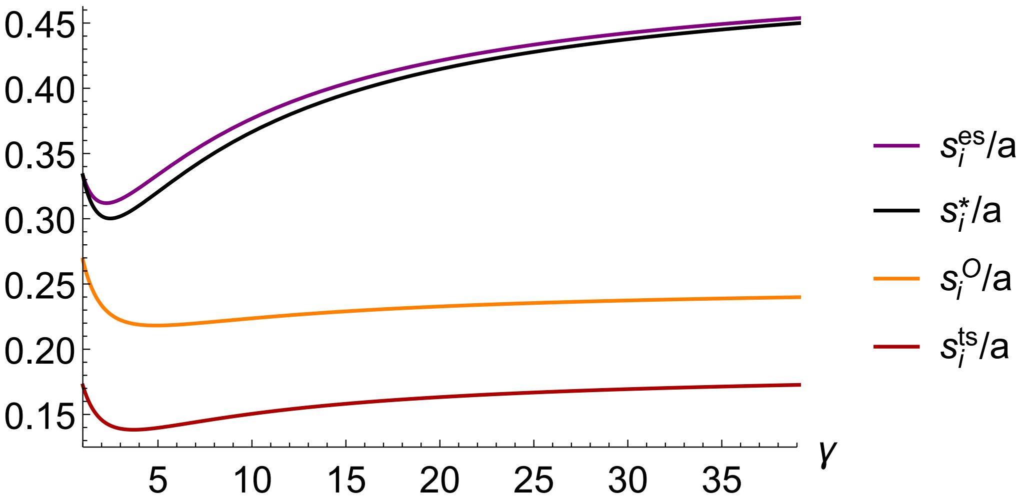

Figure 3: All production subsidies

Note: Vertical axis is subsidy rate

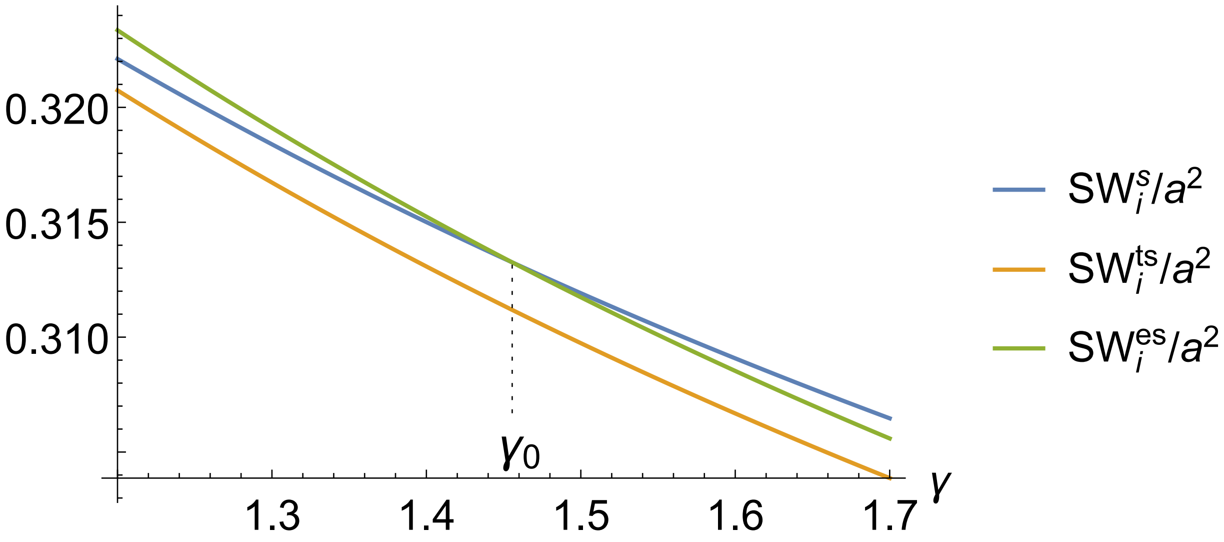

Figure 4: Welfares: Production subsidy standard

Note: Vertical axis is welfare level

Online Appendix of "Trade warfare revisited: Trade and industrial policies when exporting and non-exporting firms coexist"
(Not for Publication)

by Kazuhiro Takauchi, Hajime Sugeta, and Tomomichi Mizuno

This appendix provides (I) the second order (sufficient) conditions for welfare maximization in Section 3.1, (II) the stability condition of the equilibrium in three-policy game (Section 3.1), (III) the proof of Proposition 8, and (IV) production and export subsidies in many-firm case.

I. The second-order condition (Section 3.1)

To simplify calculation, we set the following:

$$q_i^d \equiv z_i, \quad q_{ii}^e \equiv y_i, \text{ and } q_{ij}^e \equiv x_i \quad (i \neq j \text{ and } i, j = A, B).$$

Let total sales in country i as

$$D_i \equiv x_j + y_i + z_i.$$

Similarly, let agregate production in country i as

$$Q_i \equiv x_i + y_i + z_i.$$

From the first-order conditions for welfare maximization, we obtain

$$\frac{\partial SW_i}{\partial s_i} = D_i \frac{\partial D_i}{\partial s_i} + (\gamma+2) z_i \frac{\partial z_i}{\partial s_i} + 3q_i^e \frac{\partial q_i^e}{\partial s_i} - 2\frac{\partial x_i}{\partial s_i} y_i - 2x_i \frac{\partial y_i}{\partial s_i} - Q_i - s_i \frac{\partial Q_i}{\partial s_i} - e_i \frac{\partial x_i}{\partial s_i} + t_i \frac{\partial x_j}{\partial s_i} = 0,$$

$$\frac{\partial SW_i}{\partial t_i} = D_i \frac{\partial D_i}{\partial t_i} + (\gamma+2) z_i \frac{\partial z_i}{\partial t_i} + 3q_i^e \frac{\partial q_i^e}{\partial t_i} - 2\frac{\partial x_i}{\partial t_i} y_i - 2x_i \frac{\partial y_i}{\partial t_i} - s_i \frac{\partial Q_i}{\partial t_i} - e_i \frac{\partial x_i}{\partial t_i} + x_j + t_i \frac{\partial x_j}{\partial t_i} = 0,$$

$$\frac{\partial SW_i}{\partial e_i} = D_i \frac{\partial D_i}{\partial e_i} + (\gamma+2) z_i \frac{\partial z_i}{\partial e_i} + 3q_i^e \frac{\partial q_i^e}{\partial e_i} - 2\frac{\partial x_i}{\partial e_i} y_i - 2x_i \frac{\partial y_i}{\partial e_i} - s_i \frac{\partial Q_i}{\partial e_i} - x_i - e_i \frac{\partial x_i}{\partial e_i} + t_i \frac{\partial x_j}{\partial e_i} = 0.$$

We derive the second-order conditions as follows:

$$\begin{aligned}
h_{11} &\equiv \frac{\partial^2 SW_i}{\partial s_i^2} = \left(\frac{\partial D_i}{\partial s_i}\right)^2 + (\gamma+2)\left(\frac{\partial z_i}{\partial s_i}\right)^2 + 3\left(\frac{\partial q_i^e}{\partial s_i}\right)^2 - 4\frac{\partial x_i}{\partial s_i}\frac{\partial y_i}{\partial s_i} - 2\frac{\partial Q_i}{\partial s_i} \\
&= \left[\frac{14\gamma+3\gamma^2+14}{(3\gamma+4)(5\gamma+8)}\right]^2 + (\gamma+2)\left[\frac{6(2\gamma+3)}{(5\gamma+8)(3\gamma+4)}\right]^2 + 3\left[\frac{8\gamma+11}{3(5\gamma+8)}\right]^2 \\
&\quad - 4\frac{2\left(20\gamma+6\gamma^2+17\right)}{3(5\gamma+8)(3\gamma+4)}\frac{25\gamma+12\gamma^2+10}{3(5\gamma+8)(3\gamma+4)} - 2\frac{101\gamma+24\gamma^2+98}{3(3\gamma+4)(5\gamma+8)} \\
&= -\frac{18884\gamma+18193\gamma^2+7134\gamma^3+927\gamma^4+6772}{9(3\gamma+4)^2(5\gamma+8)^2}, \\
h_{12} &\equiv \frac{\partial^2 SW_i}{\partial s_i \partial t_i} = \frac{\partial D_i}{\partial t_i}\frac{\partial D_i}{\partial s_i} + (\gamma+2)\frac{\partial z_i}{\partial t_i}\frac{\partial z_i}{\partial s_i} + 3\frac{\partial q_i^e}{\partial t_i}\frac{\partial q_i^e}{\partial s_i} - 2\frac{\partial x_i}{\partial s_i}\frac{\partial y_i}{\partial t_i} - 2\frac{\partial x_i}{\partial t_i}\frac{\partial y_i}{\partial s_i} - \frac{\partial Q_i}{\partial t_i} + \frac{\partial x_j}{\partial s_i} \\
&= \frac{-2(\gamma+1)(2\gamma+3)}{(3\gamma+4)(5\gamma+8)}\frac{14\gamma+3\gamma^2+14}{(3\gamma+4)(5\gamma+8)} + (\gamma+2)\frac{2(2\gamma+3)}{(5\gamma+8)(3\gamma+4)}\frac{6(2\gamma+3)}{(5\gamma+8)(3\gamma+4)} \\
&\quad + 3\frac{\gamma+1}{3(5\gamma+8)}\frac{8\gamma+11}{3(5\gamma+8)} - 2\frac{2\left(20\gamma+6\gamma^2+17\right)}{3(5\gamma+8)(3\gamma+4)}\frac{(\gamma+1)(9\gamma+14)}{3(5\gamma+8)(3\gamma+4)} \\
&\quad - 2\frac{-2(3\gamma+5)(\gamma+1)}{3(5\gamma+8)(3\gamma+4)}\frac{25\gamma+12\gamma^2+10}{3(5\gamma+8)(3\gamma+4)} - \frac{19\gamma+3\gamma^2+22}{3(3\gamma+4)(5\gamma+8)} + \frac{-\left(19\gamma+3\gamma^2+22\right)}{3(5\gamma+8)(3\gamma+4)} \\
&= -\frac{(\gamma+1)\left(4808\gamma+2109\gamma^2+234\gamma^3+3260\right)}{9(3\gamma+4)^2(5\gamma+8)^2}, \\
h_{13} &\equiv \frac{\partial^2 SW_i}{\partial s_i \partial e_i} = \frac{\partial D_i}{\partial e_i}\frac{\partial D_i}{\partial s_i} + (\gamma+2)\frac{\partial z_i}{\partial e_i}\frac{\partial z_i}{\partial s_i} + 3\frac{\partial q_i^e}{\partial e_i}\frac{\partial q_i^e}{\partial s_i} - 2\frac{\partial x_i}{\partial s_i}\frac{\partial y_i}{\partial e_i} - 2\frac{\partial x_i}{\partial e_i}\frac{\partial y_i}{\partial s_i} - \frac{\partial Q_i}{\partial e_i} - \frac{\partial x_i}{\partial s_i} \\
&= \frac{-(\gamma+1)(\gamma+2)}{(3\gamma+4)(5\gamma+8)}\frac{14\gamma+3\gamma^2+14}{(3\gamma+4)(5\gamma+8)} + (\gamma+2)\frac{(\gamma+2)}{(5\gamma+8)(3\gamma+4)}\frac{6(2\gamma+3)}{(5\gamma+8)(3\gamma+4)} \\
&\quad + 3\frac{(4\gamma+7)}{3(5\gamma+8)}\frac{8\gamma+11}{3(5\gamma+8)} - 2\frac{2\left(20\gamma+6\gamma^2+17\right)}{3(5\gamma+8)(3\gamma+4)}\frac{-\left(28\gamma+9\gamma^2+22\right)}{3(5\gamma+8)(3\gamma+4)} \\
&\quad - 2\frac{(3\gamma+5)(7\gamma+10)}{3(5\gamma+8)(3\gamma+4)}\frac{25\gamma+12\gamma^2+10}{3(5\gamma+8)(3\gamma+4)} - \frac{2\left(20\gamma+6\gamma^2+17\right)}{3(3\gamma+4)(5\gamma+8)} - \frac{2\left(20\gamma+6\gamma^2+17\right)}{3(5\gamma+8)(3\gamma+4)} \\
&= -\frac{5998\gamma+6563\gamma^2+3081\gamma^3+531\gamma^4+1940}{9(3\gamma+4)^2(5\gamma+8)^2},
\end{aligned}$$

$$
\begin{aligned}
h_{21} &\equiv \frac{\partial^2 SW_i}{\partial t_i \partial s_i} = \frac{\partial D_i}{\partial s_i}\frac{\partial D_i}{\partial t_i} + (\gamma+2)\frac{\partial z_i}{\partial s_i}\frac{\partial z_i}{\partial t_i} + 3\frac{\partial q_i^e}{\partial s_i}\frac{\partial q_i^e}{\partial t_i} - 2\frac{\partial x_i}{\partial t_i}\frac{\partial y_i}{\partial s_i} - 2\frac{\partial x_i}{\partial s_i}\frac{\partial y_i}{\partial t_i} - \frac{\partial Q_i}{\partial t_i} + \frac{\partial x_j}{\partial s_i} = h_{12}, \\
h_{22} &\equiv \frac{\partial^2 SW_i}{\partial t_i^2} = \left(\frac{\partial D_i}{\partial t_i}\right)^2 + (\gamma+2)\left(\frac{\partial z_i}{\partial t_i}\right)^2 + 3\left(\frac{\partial q_i^e}{\partial t_i}\right)^2 - 4\frac{\partial x_i}{\partial t_i}\frac{\partial y_i}{\partial t_i} + 2\frac{\partial x_j}{\partial t_i} \\
&= \left[\frac{-2(\gamma+1)(2\gamma+3)}{(3\gamma+4)(5\gamma+8)}\right]^2 + (\gamma+2)\left[\frac{2(2\gamma+3)}{(5\gamma+8)(3\gamma+4)}\right]^2 + 3\left[\frac{\gamma+1}{3(5\gamma+8)}\right]^2 \\
&\quad - 4\frac{-2(3\gamma+5)(\gamma+1)}{3(5\gamma+8)(3\gamma+4)}\frac{(\gamma+1)(9\gamma+14)}{3(5\gamma+8)(3\gamma+4)} + 2\frac{-(3\gamma+5)(7\gamma+10)}{3(5\gamma+8)(3\gamma+4)} \\
&= -\frac{21428\gamma + 21253\gamma^2 + 9276\gamma^3 + 1503\gamma^4 + 8020}{9(3\gamma+4)^2(5\gamma+8)^2}, \\
h_{23} &\equiv \frac{\partial^2 SW_i}{\partial t_i \partial e_i} = \frac{\partial D_i}{\partial e_i}\frac{\partial D_i}{\partial t_i} + (\gamma+2)\frac{\partial z_i}{\partial e_i}\frac{\partial z_i}{\partial t_i} + 3\frac{\partial q_i^e}{\partial e_i}\frac{\partial q_i^e}{\partial t_i} - 2\frac{\partial x_i}{\partial t_i}\frac{\partial y_i}{\partial e_i} - 2\frac{\partial x_i}{\partial e_i}\frac{\partial y_i}{\partial t_i} - \frac{\partial x_i}{\partial t_i} + \frac{\partial x_j}{\partial e_i} \\
&= \frac{-(\gamma+1)(\gamma+2)}{(3\gamma+4)(5\gamma+8)}\frac{-2(\gamma+1)(2\gamma+3)}{(3\gamma+4)(5\gamma+8)} + (\gamma+2)\frac{(\gamma+2)}{(5\gamma+8)(3\gamma+4)}\frac{2(2\gamma+3)}{(5\gamma+8)(3\gamma+4)} \\
&\quad + 3\frac{(4\gamma+7)}{3(5\gamma+8)}\frac{\gamma+1}{3(5\gamma+8)} - 2\frac{-2(3\gamma+5)(\gamma+1)}{3(5\gamma+8)(3\gamma+4)}\frac{-\left(28\gamma+9\gamma^2+22\right)}{3(5\gamma+8)(3\gamma+4)} \\
&\quad - 2\frac{(3\gamma+5)(7\gamma+10)}{3(5\gamma+8)(3\gamma+4)}\frac{(\gamma+1)(9\gamma+14)}{3(5\gamma+8)(3\gamma+4)} - \frac{-2(3\gamma+5)(\gamma+1)}{3(5\gamma+8)(3\gamma+4)} + \frac{2(3\gamma+5)(\gamma+1)}{3(5\gamma+8)(3\gamma+4)} \\
&= \frac{2062\gamma + 2225\gamma^2 + 1083\gamma^3 + 198\gamma^4 + 740}{9(3\gamma+4)^2(5\gamma+8)^2}
\end{aligned}
$$

$$
\begin{aligned}
h_{31} &\equiv \frac{\partial^2 SW_i}{\partial e_i \partial s_i} = \frac{\partial D_i}{\partial s_i}\frac{\partial D_i}{\partial e_i} + (\gamma+2)\frac{\partial z_i}{\partial s_i}\frac{\partial z_i}{\partial e_i} + 3\frac{\partial q_i^e}{\partial s_i}\frac{\partial q_i^e}{\partial e_i} - 2\frac{\partial x_i}{\partial e_i}\frac{\partial y_i}{\partial s_i} - 2\frac{\partial x_i}{\partial s_i}\frac{\partial y_i}{\partial e_i} - \frac{\partial Q_i}{\partial e_i} - \frac{\partial x_i}{\partial s_i} = h_{13}, \\
h_{32} &\equiv \frac{\partial^2 SW_i}{\partial e_i \partial t_i} = \frac{\partial D_i}{\partial t_i}\frac{\partial D_i}{\partial e_i} + (\gamma+2)\frac{\partial z_i}{\partial t_i}\frac{\partial z_i}{\partial e_i} + 3\frac{\partial q_i^e}{\partial t_i}\frac{\partial q_i^e}{\partial e_i} - 2\frac{\partial x_i}{\partial e_i}\frac{\partial y_i}{\partial t_i} - 2\frac{\partial x_i}{\partial t_i}\frac{\partial y_i}{\partial e_i} - \frac{\partial x_i}{\partial t_i} + \frac{\partial x_j}{\partial e_i} = h_{23}, \\
h_{33} &\equiv \frac{\partial^2 SW_i}{\partial e_i^2} = \left(\frac{\partial D_i}{\partial e_i}\right)^2 + (\gamma+2)\left(\frac{\partial z_i}{\partial e_i}\right)^2 + 3\left(\frac{\partial q_i^e}{\partial e_i}\right)^2 - 4\frac{\partial x_i}{\partial e_i}\frac{\partial y_i}{\partial e_i} - 2\frac{\partial x_i}{\partial e_i} \\
&= \left[\frac{-(\gamma+1)(\gamma+2)}{(3\gamma+4)(5\gamma+8)}\right]^2 + (\gamma+2)\left[\frac{\gamma+2}{(5\gamma+8)(3\gamma+4)}\right]^2 + 3\left[\frac{4\gamma+7}{3(5\gamma+8)}\right]^2 \\
&\quad - 4\frac{(3\gamma+5)(7\gamma+10)}{3(5\gamma+8)(3\gamma+4)}\frac{-\left(28\gamma+9\gamma^2+22\right)}{3(5\gamma+8)(3\gamma+4)} - 2\frac{(3\gamma+5)(7\gamma+10)}{3(5\gamma+8)(3\gamma+4)} \\
&= -\frac{\left(5188\gamma + 3282\gamma^2 + 693\gamma^3 + 2740\right)(\gamma+1)}{9(3\gamma+4)^2(5\gamma+8)^2}
\end{aligned}
$$

The Hessian matrix of SW_i is thus expressed as $H \equiv [h_{ij}]$ and its leading principal minors have the following alternating sign patterns:

$$
\begin{aligned}
&|H| \\
= &\begin{vmatrix} -\frac{18884\gamma+18193\gamma^2+7134\gamma^3+927\gamma^4+6772}{9(3\gamma+4)^2(5\gamma+8)^2} & -\frac{(\gamma+1)(4808\gamma+2109\gamma^2+234\gamma^3+3260)}{9(3\gamma+4)^2(5\gamma+8)^2} & -\frac{5998\gamma+6563\gamma^2+3081\gamma^3+531\gamma^4+1940}{9(3\gamma+4)^2(5\gamma+8)^2} \\ -\frac{(\gamma+1)\left(4808\gamma+2109\gamma^2+234\gamma^3+3260\right)}{9(3\gamma+4)^2(5\gamma+8)^2} & -\frac{21428\gamma+21253\gamma^2+9276\gamma^3+1503\gamma^4+8020}{9(3\gamma+4)^2(5\gamma+8)^2} & \frac{2062\gamma+2225\gamma^2+1083\gamma^3+198\gamma^4+740}{9(3\gamma+4)^2(5\gamma+8)^2} \\ -\frac{5998\gamma+6563\gamma^2+3081\gamma^3+531\gamma^4+1940}{9(3\gamma+4)^2(5\gamma+8)^2} & \frac{2062\gamma+2225\gamma^2+1083\gamma^3+198\gamma^4+740}{9(3\gamma+4)^2(5\gamma+8)^2} & -\frac{(5188\gamma+3282\gamma^2+693\gamma^3+2740)(\gamma+1)}{9(3\gamma+4)^2(5\gamma+8)^2} \end{vmatrix} \\
= &-\frac{(3\gamma+5)(405\gamma+251\gamma^2+34\gamma^3+180)}{9\left(3\gamma+4\right)^2\left(5\gamma+8\right)^2} < 0
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} &= \begin{vmatrix} -\frac{18884\gamma+18193\gamma^2+7134\gamma^3+927\gamma^4+6772}{9(3\gamma+4)^2(5\gamma+8)^2} & -\frac{(\gamma+1)\left(4808\gamma+2109\gamma^2+234\gamma^3+3260\right)}{9(3\gamma+4)^2(5\gamma+8)^2} \\ -\frac{(\gamma+1)\left(4808\gamma+2109\gamma^2+234\gamma^3+3260\right)}{9(3\gamma+4)^2(5\gamma+8)^2} & -\frac{21428\gamma+21253\gamma^2+9276\gamma^3+1503\gamma^4+8020}{9(3\gamma+4)^2(5\gamma+8)^2} \end{vmatrix} \\
&= \frac{13436\gamma+13045\gamma^2+5122\gamma^3+661\gamma^4+4740}{9\left(3\gamma+4\right)^2\left(5\gamma+8\right)^2} > 0, \\
h_{11} &= -\frac{18884\gamma+18193\gamma^2+7134\gamma^3+927\gamma^4+6772}{9\left(3\gamma+4\right)^2\left(5\gamma+8\right)^2} < 0.
\end{aligned}
$$

II. The stability condition (Section 3.1)

To consider the stability of the Nash equilibrium outcome of the 1st-stage policy game in three policy instruments case, we consider the following dynamical system:

$$
\begin{cases} \dot{s}_i = k_1 \dfrac{\partial SW_i}{\partial s_i}, \\ \dot{t}_i = k_2 \dfrac{\partial SW_i}{\partial t_i}, \\ \dot{e}_i = k_3 \dfrac{\partial SW_i}{\partial e_i}, \end{cases}
$$

where k_i $(i = 1, 2, 3)$ is the coefficient of adjustment speed.

The stationary point of this system satisfying $\frac{\partial SW_i}{\partial s_i} = \frac{\partial SW_i}{\partial t_i} = \frac{\partial SW_i}{\partial e_i} = 0$ gives the Nash equilibrium policy instruments. Let J denote the Jacobian matrix of the above dynamical system and its determinants can be expressed as

$$|J| \equiv k_1 k_2 k_3 \begin{vmatrix} \frac{\partial^2 SW_i}{\partial s_i^2} & \frac{\partial^2 SW_i}{\partial s_i \partial t_i} & \frac{\partial^2 SW_i}{\partial s_i \partial e_i} \\ \frac{\partial^2 SW_i}{\partial t_i \partial s_i} & \frac{\partial^2 SW_i}{\partial t_i^2} & \frac{\partial^2 SW_i}{\partial t_i \partial e_i} \\ \frac{\partial SW_i}{\partial e_i \partial s_i} & \frac{\partial SW_i}{\partial e_i \partial t_i} & \frac{\partial^2 SW_i}{\partial e_i^2} \end{vmatrix}.$$

Define the matrix $G \equiv [g_{ij}]$ as

$$\begin{aligned}
g_{11} &= \frac{\partial^2 SW_i}{\partial s_i^2} \times 3(3\gamma+4)(5\gamma+8)^2 = -\left(1042\gamma + 593\gamma^2 + 96\gamma^3 + 554\right) \\
g_{12} &= \frac{\partial^2 SW_i}{\partial s_i \partial t_i} \times 3(3\gamma+4)(5\gamma+8)^2 = -\left(580\gamma + 258\gamma^2 + 27\gamma^3 + 382\right) \\
g_{13} &= \frac{\partial^2 SW_i}{\partial s_i \partial e_i} \times 3(3\gamma+4)(5\gamma+8)^2 = -\left(172\gamma + 157\gamma^2 + 48\gamma^3 + 66\right) \\
g_{21} &= \frac{\partial^2 SW_i}{\partial t_i \partial s_i} \times 3(3\gamma+4)(5\gamma+8)^2 = 2(\gamma+1)\left(9\gamma^2 - 2\gamma - 23\right) \\
g_{22} &= \frac{\partial^2 SW_i}{\partial t_i^2} \times 3(3\gamma+4)(5\gamma+8)^2 = -\left(1232\gamma + 771\gamma^2 + 159\gamma^3 + 650\right) \\
g_{23} &= \frac{\partial^2 SW_i}{\partial t_i \partial e_i} \times 3(3\gamma+4)(5\gamma+8)^2 = 2(2\gamma+3)\left(70\gamma + 21\gamma^2 + 55\right) \\
g_{31} &= \frac{\partial SW_i}{\partial e_i \partial s_i} \times 3(3\gamma+4)(5\gamma+8)^2 = -\left(479\gamma + 301\gamma^2 + 63\gamma^3 + 250\right) \\
g_{32} &= \frac{\partial SW_i}{\partial e_i \partial t_i} \times 3(3\gamma+4)(5\gamma+8)^2 = -\left(155\gamma + 63\gamma^2 + 6\gamma^3 + 110\right) \\
g_{33} &= \frac{\partial^2 SW_i}{\partial e_i^2} \times 3(3\gamma+4)(5\gamma+8)^2 = -\left(437\gamma + 302\gamma^2 + 69\gamma^3 + 210\right)
\end{aligned}$$

By setting $k_1 k_2 k_3 = \{3(3\gamma+4)(5\gamma+8)^2\}^3$, we regard G as the Jacobian matrix instead of J since

$$\begin{aligned}
& |G| \\
= \; & \{3(3\gamma+4)(5\gamma+8)^2\}^3 \begin{vmatrix} \frac{\partial^2 SW_i}{\partial s_i^2} & \frac{\partial^2 SW_i}{\partial s_i \partial t_i} & \frac{\partial^2 SW_i}{\partial s_i \partial e_i} \\ \frac{\partial^2 SW_i}{\partial t_i \partial s_i} & \frac{\partial^2 SW_i}{\partial t_i^2} & \frac{\partial^2 SW_i}{\partial t_i \partial e_i} \\ \frac{\partial SW_i}{\partial e_i \partial s_i} & \frac{\partial SW_i}{\partial e_i \partial t_i} & \frac{\partial^2 SW_i}{\partial e_i^2} \end{vmatrix} = k_1 k_2 k_3 \begin{vmatrix} \frac{\partial^2 SW_i}{\partial s_i^2} & \frac{\partial^2 SW_i}{\partial s_i \partial t_i} & \frac{\partial^2 SW_i}{\partial s_i \partial e_i} \\ \frac{\partial^2 SW_i}{\partial t_i \partial s_i} & \frac{\partial^2 SW_i}{\partial t_i^2} & \frac{\partial^2 SW_i}{\partial t_i \partial e_i} \\ \frac{\partial SW_i}{\partial e_i \partial s_i} & \frac{\partial SW_i}{\partial e_i \partial t_i} & \frac{\partial^2 SW_i}{\partial e_i^2} \end{vmatrix} \\
= \; & |J|.
\end{aligned}$$

The leading principal minors of G has the following alternating sign patterns:

$$\begin{aligned}
&|G| \\
= &\begin{vmatrix} -\left(1042\gamma+593\gamma^2+96\gamma^3+554\right) & -\left(580\gamma+258\gamma^2+27\gamma^3+382\right) & -\left(172\gamma+157\gamma^2+48\gamma^3+66\right) \\ 2\left(\gamma+1\right)\left(9\gamma^2-2\gamma-23\right) & -\left(1232\gamma+771\gamma^2+159\gamma^3+650\right) & 2\left(2\gamma+3\right)\left(70\gamma+21\gamma^2+55\right) \\ -\left(479\gamma+301\gamma^2+63\gamma^3+250\right) & -\left(155\gamma+63\gamma^2+6\gamma^3+110\right) & -\left(437\gamma+302\gamma^2+69\gamma^3+210\right) \end{vmatrix} \\
= &-9\left(3\gamma+4\right)^2\left(5\gamma+8\right)^4\left(159\gamma+82\gamma^2+10\gamma^3+85\right) < 0,
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} &= \begin{vmatrix} -\left(1042\gamma+593\gamma^2+96\gamma^3+554\right) & -\left(580\gamma+258\gamma^2+27\gamma^3+382\right) \\ 2\left(\gamma+1\right)\left(9\gamma^2-2\gamma-23\right) & -\left(1232\gamma+771\gamma^2+159\gamma^3+650\right) \end{vmatrix} \\
&= 3\left(3\gamma+4\right)\left(819\gamma+453\gamma^2+70\gamma^3+446\right)\left(5\gamma+8\right)^2 > 0, \\
g_{11} &= -\left(1042\gamma+593\gamma^2+96\gamma^3+554\right) < 0.
\end{aligned}$$

The characteristic polynomial of the Jacobian matrix G is $\begin{vmatrix} g_{11}-\lambda & g_{12} & g_{13} \\ g_{21} & g_{22}-\lambda & g_{23} \\ g_{31} & g_{32} & g_{33}-\lambda \end{vmatrix} = 0,$ which becomes

$$\begin{vmatrix} -(1042\gamma+593\gamma^2+96\gamma^3+554)-\lambda & -(580\gamma+258\gamma^2+27\gamma^3+382) & -(172\gamma+157\gamma^2+48\gamma^3+66) \\ 2(\gamma+1)(9\gamma^2-2\gamma-23) & -(1232\gamma+771\gamma^2+159\gamma^3+650)-\lambda & 2(2\gamma+3)(70\gamma+21\gamma^2+55) \\ -(479\gamma+301\gamma^2+63\gamma^3+250) & -(155\gamma+63\gamma^2+6\gamma^3+110) & -(437\gamma+302\gamma^2+69\gamma^3+210)-\lambda \end{vmatrix}$$

$$= -a_3 - a_2\lambda - a_1\lambda^2 - \lambda^3 = 0,$$

where

$$\begin{aligned}
a_1 &\equiv 2711\gamma+1666\gamma^2+324\gamma^3+1414 \\
a_2 &\equiv 3\left(3\gamma+4\right)\left(5\gamma+8\right)^2\left(1477\gamma+836\gamma^2+137\gamma^3+801\right) \\
a_3 &\equiv 9\left(3\gamma+4\right)^2\left(5\gamma+8\right)^4\left(159\gamma+82\gamma^2+10\gamma^3+85\right) = -\left|G\right|.
\end{aligned}$$

According to the Routh-Hurwitz stability criteria,[1] all the roots of the characteristic polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

[1] See, for example, Samuelson (1983), pp. 329–435.

have negative real parts if and only if

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0, \qquad D \equiv \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0.$$

We need to confirm the sign of D:

$$\begin{aligned} D &\equiv a_1 a_2 - a_3 \\ &= \left(2711\gamma + 1666\gamma^2 + 324\gamma^3 + 1414\right)\left[3(3\gamma+4)(5\gamma+8)^2(1477\gamma + 836\gamma^2 + 137\gamma^3 + 801)\right] \\ &\quad -9(3\gamma+4)^2(5\gamma+8)^4(159\gamma + 82\gamma^2 + 10\gamma^3 + 85) \\ &= 3(3\gamma+4)(5\gamma+8)^2 \\ &\quad \times(4007317\gamma + 6126817\gamma^2 + 4865383\gamma^3 + 2107956\gamma^4 + 470456\gamma^5 + 42138\gamma^6 + 1067334), \end{aligned}$$

which is positive.

III. Proof of Proposition 8

The social welfare in country i is

$$\begin{aligned} SW_i &\equiv CS_i + \pi_i^d + \pi_i^e - s_{ix}x_i - s_{iy}y_i - s_{iz}z_i + t_i x_j, \\ &= \frac{1}{2}D_i^2 + \left(\frac{1}{2}\gamma + 1\right) z_i^2 + \frac{3}{2}(q_i^e)^2 - 2x_i y_i - s_{ix}x_i - s_{iy}y_i - s_{iz}z_i + t_i x_j. \end{aligned}$$

The first-order conditions for welfare maximization are thus

$$\frac{\partial SW_i}{\partial s_{ix}} = D_i\frac{\partial D_i}{\partial s_{ix}} + (\gamma+2)\, z_i \frac{\partial z_i}{\partial s_{ix}} + 3q_i^e \frac{\partial q_i^e}{\partial s_{ix}} - 2\frac{\partial x_i}{\partial s_{ix}} y_i - 2x_i \frac{\partial y_i}{\partial s_{ix}} - x_i - s_{ix}\frac{\partial x_i}{\partial s_{ix}} - s_{iy}\frac{\partial y_i}{\partial s_{ix}} - s_{iz}\frac{\partial z_i}{\partial s_{ix}} + t_i \frac{\partial x_j}{\partial s_{ix}} = 0$$
$$\frac{\partial SW_i}{\partial s_{iy}} = D_i\frac{\partial D_i}{\partial s_{iy}} + (\gamma+2)\, z_i \frac{\partial z_i}{\partial s_{iy}} + 3q_i^e \frac{\partial q_i^e}{\partial s_{iy}} - 2\frac{\partial x_i}{\partial s_{iy}} y_i - 2x_i \frac{\partial y_i}{\partial s_{iy}} - s_{ix}\frac{\partial x_i}{\partial s_{iy}} - y_i - s_{iy}\frac{\partial y_i}{\partial s_{iy}} - s_{iz}\frac{\partial z_i}{\partial s_{iy}} + t_i \frac{\partial x_j}{\partial s_{iy}} = 0$$
$$\frac{\partial SW_i}{\partial s_{iz}} = D_i\frac{\partial D_i}{\partial s_{iz}} + (\gamma+2)\, z_i \frac{\partial z_i}{\partial s_{iz}} + 3q_i^e \frac{\partial q_i^e}{\partial s_{iz}} - 2\frac{\partial x_i}{\partial s_{iz}} y_i - 2x_i \frac{\partial y_i}{\partial s_{iz}} - s_{ix}\frac{\partial x_i}{\partial s_{iz}} - s_{iy}\frac{\partial y_i}{\partial s_{iz}} - z_i - s_{iz}\frac{\partial z_i}{\partial s_{iz}} + t_i \frac{\partial x_j}{\partial s_{iz}} = 0$$
$$\frac{\partial SW_i}{\partial t_i} = D_i\frac{\partial D_i}{\partial t_i} + (\gamma+2)\, z_i \frac{\partial z_i}{\partial t_i} + 3q_i^e \frac{\partial q_i^e}{\partial t_i} - 2\frac{\partial x_i}{\partial t_i} y_i - 2x_i \frac{\partial y_i}{\partial t_i} - s_{ix}\frac{\partial x_i}{\partial t_i} - s_{iy}\frac{\partial y_i}{\partial t_i} - s_{iz}\frac{\partial z_i}{\partial t_i} + x_j + t_i \frac{\partial x_j}{\partial t_i} = 0$$

We use symmetry of the model to derive the Nash equilibrium outcome of this this four-instrument policy game. Evaluating at the symmetric policy instruments $s_{ix} = s_{jx}$, $s_{iy} = s_{jy}$,

$s_{iz} = s_{jz}$, and $t_i = t_j$, we write the Nash equilibrium conditions as:

$$\begin{aligned}
\frac{\partial SW_i}{\partial s_{ix}} &= D_i \frac{\partial D_i}{\partial s_{ix}} - x_i + 3q_i^e \frac{\partial q_i^e}{\partial s_{ix}} + ((\gamma+2) z_i - s_{iz}) \frac{\partial z_i}{\partial s_{ix}} - (2y_i + s_{ix}) \frac{\partial x_i}{\partial s_{ix}} - (2x_i + s_{iy}) \frac{\partial y_i}{\partial s_{ix}} + t_i \frac{\partial x_j}{\partial s_{ix}} = 0 \\
\frac{\partial SW_i}{\partial s_{iy}} &= D_i \frac{\partial D_i}{\partial s_{iy}} - y_i + 3q_i^e \frac{\partial q_i^e}{\partial s_{iy}} + ((\gamma+2) z_i - s_{iz}) \frac{\partial z_i}{\partial s_{iy}} - (2y_i + s_{ix}) \frac{\partial x_i}{\partial s_{iy}} - (2x_i + s_{iy}) \frac{\partial y_i}{\partial s_{iy}} + t_i \frac{\partial x_j}{\partial s_{iy}} = 0 \\
\frac{\partial SW_i}{\partial s_{iz}} &= D_i \frac{\partial D_i}{\partial s_{iz}} - z_i + 3q_i^e \frac{\partial q_i^e}{\partial s_{iz}} + ((\gamma+2) z_i - s_{iz}) \frac{\partial z_i}{\partial s_{iz}} - (2y_i + s_{ix}) \frac{\partial x_i}{\partial s_{iz}} - (2x_i + s_{iy}) \frac{\partial y_i}{\partial s_{iz}} + t_i \frac{\partial x_j}{\partial s_{iz}} = 0 \\
\frac{\partial SW_i}{\partial t_i} &= D_i \frac{\partial D_i}{\partial t_i} + x_j + 3q_i^e \frac{\partial q_i^e}{\partial t_i} + ((\gamma+2) z_i - s_{iz}) \frac{\partial z_i}{\partial t_i} - (2y_i + s_{ix}) \frac{\partial x_i}{\partial t_i} - (2x_i + s_{iy}) \frac{\partial y_i}{\partial t_i} + t_i \frac{\partial x_j}{\partial t_i} = 0
\end{aligned}$$

Substituting the partial derivatives yields

$$\begin{aligned}
\frac{\partial SW_i}{\partial s_{ix}} &= D_i \frac{-(\gamma+2)(\gamma+1)}{(3\gamma+4)(5\gamma+8)} - x_i + 3q_i^e \frac{(4\gamma+7)}{3(5\gamma+8)} + ((\gamma+2) z_i - s_{iz}) \frac{(\gamma+2)}{(3\gamma+4)(5\gamma+8)} \\
&\quad - (2y_i + s_{ix}) \frac{(3\gamma+5)(7\gamma+10)}{3(3\gamma+4)(5\gamma+8)} - (2x_i + s_{iy}) \frac{-\left(28\gamma+9\gamma^2+22\right)}{3(3\gamma+4)(5\gamma+8)} + t_i \frac{2(3\gamma+5)(\gamma+1)}{3(3\gamma+4)(5\gamma+8)}, \\
\frac{\partial SW_i}{\partial s_{iy}} &= D_i \frac{2(2\gamma+3)(\gamma+1)}{(3\gamma+4)(5\gamma+8)} - y_i + 3q_i^e \frac{(4\gamma+7)}{3(5\gamma+8)} + ((\gamma+2) z_i - s_{iz}) \frac{-2(2\gamma+3)}{(3\gamma+4)(5\gamma+8)} \\
&\quad - (2y_i + s_{ix}) \frac{-\left(28\gamma+9\gamma^2+22\right)}{3(3\gamma+4)(5\gamma+8)} - (2x_i + s_{iy}) \frac{(3\gamma+5)(7\gamma+10)}{3(3\gamma+4)(5\gamma+8)} + t_i \frac{2(3\gamma+5)(\gamma+1)}{3(3\gamma+4)(5\gamma+8)}, \\
\frac{\partial SW_i}{\partial s_{iz}} &= D_i \frac{(7\gamma+10)}{(3\gamma+4)(5\gamma+8)} - z_i + 3q_i^e \frac{-3}{3(5\gamma+8)} + ((\gamma+2) z_i - s_{iz}) \frac{(15\gamma+22)}{(3\gamma+4)(5\gamma+8)} \\
&\quad - (2y_i + s_{ix}) \frac{3(\gamma+2)}{3(3\gamma+4)(5\gamma+8)} - (2x_i + s_{iy}) \frac{-6(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)} + t_i \frac{-6(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)}, \\
\frac{\partial SW_i}{\partial t_i} &= D_i \frac{-2(2\gamma+3)(\gamma+1)}{(3\gamma+4)(5\gamma+8)} + x_j + 3q_i^e \frac{(\gamma+1)}{3(5\gamma+8)} + ((\gamma+2) z_i - s_{iz}) \frac{2(2\gamma+3)}{(3\gamma+4)(5\gamma+8)} \\
&\quad - (2y_i + s_{ix}) \frac{-2(3\gamma+5)(\gamma+1)}{3(3\gamma+4)(5\gamma+8)} - (2x_i + s_{iy}) \frac{(9\gamma+14)(\gamma+1)}{3(3\gamma+4)(5\gamma+8)} + t_i \frac{-(3\gamma+5)(7\gamma+10)}{3(3\gamma+4)(5\gamma+8)}.
\end{aligned}$$

Then we define:

$$
\begin{aligned}
f_x &\equiv \frac{\partial SW_i}{\partial s_{ix}} \times 3\left(3\gamma+4\right)\left(5\gamma+8\right) \\
&= -3D_i\left(\gamma+2\right)\left(\gamma+1\right) - 3x_i\left(3\gamma+4\right)\left(5\gamma+8\right) + 3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) \\
&\quad +3\left(\gamma+2\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) - \left(2y_i+s_{ix}\right)\left(3\gamma+5\right)\left(7\gamma+10\right) \\
&\quad +\left(2x_i+s_{iy}\right)\left(28\gamma+9\gamma^2+22\right) + 2t_i\left(3\gamma+5\right)\left(\gamma+1\right), \\
f_y &\equiv \frac{\partial SW_i}{\partial s_{iy}} \times 3\left(3\gamma+4\right)\left(5\gamma+8\right) \\
&= 6D_i\left(2\gamma+3\right)\left(\gamma+1\right) - 3y_i\left(3\gamma+4\right)\left(5\gamma+8\right) + 3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) \\
&\quad -6\left(2\gamma+3\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) + \left(2y_i+s_{ix}\right)\left(28\gamma+9\gamma^2+22\right) \\
&\quad -\left(2x_i+s_{iy}\right)\left(3\gamma+5\right)\left(7\gamma+10\right) + 2t_i\left(3\gamma+5\right)\left(\gamma+1\right), \\
f_z &\equiv \frac{\partial SW_i}{\partial s_{iz}} \times 3\left(3\gamma+4\right)\left(5\gamma+8\right) \\
&= 3D_i\left(7\gamma+10\right) - 3z_i\left(3\gamma+4\right)\left(5\gamma+8\right) - 9q_i^e\left(3\gamma+4\right) + 3\left(15\gamma+22\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) \\
&\quad -3\left(2y_i+s_{ix}\right)\left(\gamma+2\right) + 6\left(2x_i+s_{iy}\right)\left(2\gamma+3\right) - 6t_i\left(2\gamma+3\right) \\
f_t &\equiv \frac{\partial SW_i}{\partial t_i} \times 3\left(3\gamma+4\right)\left(5\gamma+8\right) \\
&= -6D_i\left(2\gamma+3\right)\left(\gamma+1\right) + 3x_j\left(3\gamma+4\right)\left(5\gamma+8\right) + 3q_i^e\left(\gamma+1\right)\left(3\gamma+4\right) \\
&\quad +6\left(2\gamma+3\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) + 2\left(2y_i+s_{ix}\right)\left(3\gamma+5\right)\left(\gamma+1\right) \\
&\quad -\left(2x_i+s_{iy}\right)\left(9\gamma+14\right)\left(\gamma+1\right) - t_i\left(3\gamma+5\right)\left(7\gamma+10\right)
\end{aligned}
$$

From the first-order conditions, $f_x = 0$, $f_y = 0$, $f_z = 0$, and $f_t = 0$, it follows that

$$
\begin{aligned}
&3\left(\gamma+2\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) - \left(2y_i+s_{ix}\right)\left(3\gamma+5\right)\left(7\gamma+10\right) + \left(2x_i+s_{iy}\right)\left(28\gamma+9\gamma^2+22\right) + 2t_i\left(3\gamma+5\right)\left(\gamma+1\right) \\
&\quad = 3D_i\left(\gamma+2\right)\left(\gamma+1\right) + 3x_i\left(3\gamma+4\right)\left(5\gamma+8\right) - 3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) \\
&-6\left(2\gamma+3\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) + \left(2y_i+s_{ix}\right)\left(28\gamma+9\gamma^2+22\right) - \left(2x_i+s_{iy}\right)\left(3\gamma+5\right)\left(7\gamma+10\right) + 2t_i\left(3\gamma+5\right)\left(\gamma+1\right) \\
&\quad = -6D_i\left(2\gamma+3\right)\left(\gamma+1\right) + 3y_i\left(3\gamma+4\right)\left(5\gamma+8\right) - 3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) \\
&3\left(15\gamma+22\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) - 3\left(2y_i+s_{ix}\right)\left(\gamma+2\right) + 6\left(2x_i+s_{iy}\right)\left(2\gamma+3\right) - 6t_i\left(2\gamma+3\right) \\
&\quad = -3D_i\left(7\gamma+10\right) + 3z_i\left(3\gamma+4\right)\left(5\gamma+8\right) + 9q_i^e\left(3\gamma+4\right) \\
&6\left(2\gamma+3\right)\left(\left(\gamma+2\right)z_i - s_{iz}\right) + 2\left(2y_i+s_{ix}\right)\left(3\gamma+5\right)\left(\gamma+1\right) - \left(2x_i+s_{iy}\right)\left(9\gamma+14\right)\left(\gamma+1\right) - t_i\left(3\gamma+5\right)\left(7\gamma+10\right) \\
&\quad = 6D_i\left(2\gamma+3\right)\left(\gamma+1\right) - 3x_j\left(3\gamma+4\right)\left(5\gamma+8\right) - 3q_i^e\left(\gamma+1\right)\left(3\gamma+4\right)
\end{aligned}
$$

$$3\left(\gamma+2\right)\left(\left(\gamma+2\right)z_i-s_{iz}\right)-\left(2y_i+s_{ix}\right)\left(3\gamma+5\right)\left(7\gamma+10\right)+\left(2x_i+s_{iy}\right)\left(28\gamma+9\gamma^2+22\right)+2t_i\left(3\gamma+5\right)\left(\gamma+1\right)$$
$$=3D_i\left(\gamma+2\right)\left(\gamma+1\right)+3x_i\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right)$$
$$-6\left(2\gamma+3\right)\left(\left(\gamma+2\right)z_i-s_{iz}\right)+\left(2y_i+s_{ix}\right)\left(28\gamma+9\gamma^2+22\right)-\left(2x_i+s_{iy}\right)\left(3\gamma+5\right)\left(7\gamma+10\right)+2t_i\left(3\gamma+5\right)\left(\gamma+1\right)$$
$$=-6D_i\left(2\gamma+3\right)\left(\gamma+1\right)+3y_i\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right)$$
$$3\left(15\gamma+22\right)\left(\left(\gamma+2\right)z_i-s_{iz}\right)-3\left(2y_i+s_{ix}\right)\left(\gamma+2\right)+6\left(2x_i+s_{iy}\right)\left(2\gamma+3\right)-6t_i\left(2\gamma+3\right)$$
$$=-3D_i\left(7\gamma+10\right)+3z_i\left(3\gamma+4\right)\left(5\gamma+8\right)+9q_i^e\left(3\gamma+4\right)$$
$$6\left(2\gamma+3\right)\left(\left(\gamma+2\right)z_i-s_{iz}\right)+2\left(2y_i+s_{ix}\right)\left(3\gamma+5\right)\left(\gamma+1\right)-\left(2x_i+s_{iy}\right)\left(9\gamma+14\right)\left(\gamma+1\right)-t_i\left(3\gamma+5\right)\left(7\gamma+10\right)$$
$$=6D_i\left(2\gamma+3\right)\left(\gamma+1\right)-3x_j\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(\gamma+1\right)\left(3\gamma+4\right)$$

$$\begin{bmatrix} 3\left(\gamma+2\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) & \left(28\gamma+9\gamma^2+22\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) \\ -6\left(2\gamma+3\right) & \left(28\gamma+9\gamma^2+22\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) \\ \left(15\gamma+22\right) & -\left(\gamma+2\right) & 2\left(2\gamma+3\right) & -2\left(2\gamma+3\right) \\ 6\left(2\gamma+3\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) & -\left(9\gamma+14\right)\left(\gamma+1\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) \end{bmatrix}\begin{bmatrix} (\gamma+2)z_i-s_{iz} \\ 2y_i+s_{ix} \\ 2x_i+s_{iy} \\ t_i \end{bmatrix}$$

$$=\begin{bmatrix} 3D_i\left(\gamma+2\right)\left(\gamma+1\right)+3x_i\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) \\ -6D_i\left(2\gamma+3\right)\left(\gamma+1\right)+3y_i\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) \\ -D_i\left(7\gamma+10\right)+z_i\left(3\gamma+4\right)\left(5\gamma+8\right)+3q_i^e\left(3\gamma+4\right) \\ 6D_i\left(2\gamma+3\right)\left(\gamma+1\right)-3x_j\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(\gamma+1\right)\left(3\gamma+4\right) \end{bmatrix}$$

Let H denote the determinant of the coefficient matrix in the above matrix equation. Then applying Cramer's rule gives

$$H\left(\left(\gamma+2\right)z_i-s_{iz}\right)$$
$$=\begin{vmatrix} 3D_i\left(\gamma+2\right)\left(\gamma+1\right)+3x_i\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) & \left(28\gamma+9\gamma^2+22\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) \\ -6D_i\left(2\gamma+3\right)\left(\gamma+1\right)+3y_i\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(4\gamma+7\right)\left(3\gamma+4\right) & \left(28\gamma+9\gamma^2+22\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) \\ -D_i\left(7\gamma+10\right)+z_i\left(3\gamma+4\right)\left(5\gamma+8\right)+3q_i^e\left(3\gamma+4\right) & -\left(\gamma+2\right) & 2\left(2\gamma+3\right) & -2\left(2\gamma+3\right) \\ 6D_i\left(2\gamma+3\right)\left(\gamma+1\right)-3x_j\left(3\gamma+4\right)\left(5\gamma+8\right)-3q_i^e\left(\gamma+1\right)\left(3\gamma+4\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) & -\left(9\gamma+14\right)\left(\gamma+1\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) \end{vmatrix},$$

where

$$\begin{aligned} H &\equiv \begin{vmatrix} 3\left(\gamma+2\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) & \left(28\gamma+9\gamma^2+22\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) \\ -6\left(2\gamma+3\right) & \left(28\gamma+9\gamma^2+22\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) \\ \left(15\gamma+22\right) & -\left(\gamma+2\right) & 2\left(2\gamma+3\right) & -2\left(2\gamma+3\right) \\ 6\left(2\gamma+3\right) & 2\left(3\gamma+5\right)\left(\gamma+1\right) & -\left(9\gamma+14\right)\left(\gamma+1\right) & -\left(3\gamma+5\right)\left(7\gamma+10\right) \end{vmatrix} \\ &= -9\left(3\gamma+5\right)\left(4\gamma+5\right)\left(3\gamma+4\right)^2\left(5\gamma+8\right)^3<0. \end{aligned}$$

Production subsidy granted to the non-exporting firm:

$$
\begin{aligned}
& H\left(\left(\gamma+2\right)z_i - s_{iz}\right) \\
= & \begin{vmatrix}
3D_i(\gamma+2)(\gamma+1)+3x_i(3\gamma+4)(5\gamma+8)-3q_i^e(4\gamma+7)(3\gamma+4) & -(3\gamma+5)(7\gamma+10) & \left(28\gamma+9\gamma^2+22\right) & 2(3\gamma+5)(\gamma+1) \\
-6D_i(2\gamma+3)(\gamma+1)+3y_i(3\gamma+4)(5\gamma+8)-3q_i^e(4\gamma+7)(3\gamma+4) & \left(28\gamma+9\gamma^2+22\right) & -(3\gamma+5)(7\gamma+10) & 2(3\gamma+5)(\gamma+1) \\
-D_i(7\gamma+10)+z_i(3\gamma+4)(5\gamma+8)+3q_i^e(3\gamma+4) & -(\gamma+2) & 2(2\gamma+3) & -2(2\gamma+3) \\
6D_i(2\gamma+3)(\gamma+1)-3x_j(3\gamma+4)(5\gamma+8)-3q_i^e(\gamma+1)(3\gamma+4) & 2(3\gamma+5)(\gamma+1) & -(9\gamma+14)(\gamma+1) & -(3\gamma+5)(7\gamma+10)
\end{vmatrix} \\
= & \begin{vmatrix}
-D_i(5\gamma+8)^2+(3\gamma+4)(5\gamma+8)(3x_i+z_i(4\gamma+7)) & -(5\gamma+8)^2 & (5\gamma+8)^2 & -2(\gamma+2)(5\gamma+8) \\
-D_i(5\gamma+8)(8\gamma+11)+(3\gamma+4)(5\gamma+8)(3y_i+z_i(4\gamma+7)) & (5\gamma+8)(\gamma+1) & -(5\gamma+8)(\gamma+1) & -2(\gamma+2)(5\gamma+8) \\
-D_i(7\gamma+10)+z_i(3\gamma+4)(5\gamma+8)+3q_i^e(3\gamma+4) & -(\gamma+2) & 2(2\gamma+3) & -2(2\gamma+3) \\
D_i(5\gamma+8)(\gamma+1)+(3\gamma+4)(5\gamma+8)\left(z_i(\gamma+1)-3x_j\right) & (5\gamma+8)(\gamma+1) & -(5\gamma+8)(\gamma+1) & -(5\gamma+8)(5\gamma+7)
\end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix}
-D_i(5\gamma+8)+(3\gamma+4)(z_i(4\gamma+7)+3x_i) & -(5\gamma+8) & (5\gamma+8) & -2(\gamma+2) \\
-D_i(8\gamma+11)+(3\gamma+4)(z_i(4\gamma+7)+3y_i) & (\gamma+1) & -(\gamma+1) & -2(\gamma+2) \\
-D_i(7\gamma+10)+(3\gamma+4)\left(z_i(5\gamma+8)+3q_i^e\right) & -(\gamma+2) & 2(2\gamma+3) & -2(2\gamma+3) \\
D_i(\gamma+1)+(3\gamma+4)\left(z_i(\gamma+1)-3x_j\right) & (\gamma+1) & -(\gamma+1) & -(5\gamma+7)
\end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix}
(3\gamma+4)(z_i(4\gamma+7)+3x_i) & 0 & (5\gamma+8) & 3\gamma+4 \\
-3D_i(3\gamma+4)+(3\gamma+4)(z_i(4\gamma+7)+3y_i) & 0 & -(\gamma+1) & -(3\gamma+5) \\
-D_i(3\gamma+4)+(3\gamma+4)(z_i(5\gamma+8)+3(x_i+y_i)) & 3\gamma+4 & 2(2\gamma+3) & 0 \\
(3\gamma+4)\left(z_i(\gamma+1)-3x_j\right) & 0 & -(\gamma+1) & -2(3\gamma+4)
\end{vmatrix} \\
= & (5\gamma+8)^3(3\gamma+4)^2 \begin{vmatrix}
z_i(4\gamma+7)+3x_i & 0 & (5\gamma+8) & 3\gamma+4 \\
-3D_i+z_i(4\gamma+7)+3y_i & 0 & -(\gamma+1) & -(3\gamma+5) \\
-D_i+z_i(5\gamma+8)+3(x_i+y_i) & 1 & 2(2\gamma+3) & 0 \\
z_i(\gamma+1)-3x_j & 0 & -(\gamma+1) & -2(3\gamma+4)
\end{vmatrix} \\
= & (5\gamma+8)^3(3\gamma+4)^2 \begin{vmatrix}
z_i(4\gamma+7)+3x_i & 0 & (5\gamma+8) & 3\gamma+4 \\
-3\left(z_i+y_i+x_j\right)+z_i(4\gamma+7)+3y_i & 0 & -(\gamma+1) & -(3\gamma+5) \\
-\left(z_i+y_i+x_j\right)+z_i(5\gamma+8)+3(x_i+y_i) & 1 & 2(2\gamma+3) & 0 \\
z_i(\gamma+1)-3x_j & 0 & -(\gamma+1) & -2(3\gamma+4)
\end{vmatrix} \\
= & (5\gamma+8)^3(3\gamma+4)^2 \begin{vmatrix}
z_i(4\gamma+7)+3x_i & 0 & (5\gamma+8) & 3\gamma+4 \\
4z_i(\gamma+1)-3x_j & 0 & -(\gamma+1) & -(3\gamma+5) \\
3x_i+2y_i-x_j+z_i(5\gamma+7) & 1 & 2(2\gamma+3) & 0 \\
z_i(\gamma+1)-3x_j & 0 & -(\gamma+1) & -2(3\gamma+4)
\end{vmatrix} \\
= & (5\gamma+8)^3(3\gamma+4)^2 \begin{vmatrix}
z_i(4\gamma+7)+3x_i & 0 & (5\gamma+8) & 3\gamma+4 \\
3(\gamma+1)z_i & 0 & 0 & 3(\gamma+1) \\
3x_i+2y_i-x_j+z_i(5\gamma+7) & 1 & 2(2\gamma+3) & 0 \\
z_i(\gamma+1)-3x_j & 0 & -(\gamma+1) & -2(3\gamma+4)
\end{vmatrix} \\
= & \begin{vmatrix}
z_i(4\gamma+7)+3x_i & 0 & (5\gamma+8) & 3\gamma+4 \\
z_i & 0 & 0 & 1 \\
3x_i+2y_i-x_j+z_i(5\gamma+7) & 1 & 2(2\gamma+3) & 0 \\
z_i(\gamma+1)-3x_j & 0 & -(\gamma+1) & -2(3\gamma+4)
\end{vmatrix} \\
= & 3(\gamma+1)(5\gamma+8)^3(3\gamma+4)^2\left(3(5\gamma+8)x_j-3(\gamma+1)x_i-3(4\gamma+5)(3\gamma+5)z_i\right) \\
= & 9(\gamma+1)(5\gamma+8)^3(3\gamma+4)^2\left(x_j(5\gamma+8)-x_i(\gamma+1)-z_i(3\gamma+5)(4\gamma+5)\right)
\end{aligned}
$$

$$
\begin{aligned}
(\gamma+2)z_i - s_{iz} &= \frac{9(\gamma+1)(5\gamma+8)^3(3\gamma+4)^2\left(x_j(5\gamma+8)-z_i(3\gamma+5)(4\gamma+5)-x_i(\gamma+1)\right)}{H} \\
&= \frac{9(\gamma+1)(5\gamma+8)^3(3\gamma+4)^2\left(x_j(5\gamma+8)-z_i(3\gamma+5)(4\gamma+5)-x_i(\gamma+1)\right)}{-9(3\gamma+5)(4\gamma+5)(3\gamma+4)^2(5\gamma+8)^3} \\
&= -\frac{(\gamma+1)\left(x_j(5\gamma+8)-z_i(3\gamma+5)(4\gamma+5)-x_i(\gamma+1)\right)}{(3\gamma+5)(4\gamma+5)}
\end{aligned}
$$

Hence

$$(\gamma+2)\,z_i - s_{iz} = -\frac{(\gamma+1)\,(x_j\,(5\gamma+8) - z_i\,(3\gamma+5)\,(4\gamma+5) - x_i\,(\gamma+1))}{(3\gamma+5)\,(4\gamma+5)}$$

Solve this for s_{iz} to get

$$\begin{aligned}
s_{iz} &= (\gamma+2)\,z_i + \frac{(\gamma+1)\,(x_j\,(5\gamma+8) - z_i\,(3\gamma+5)\,(4\gamma+5) - x_i\,(\gamma+1))}{(3\gamma+5)\,(4\gamma+5)} \\
&= \frac{(3\gamma+5)\,(4\gamma+5)\,(\gamma+2)\,z_i + x_j\,(5\gamma+8)\,(\gamma+1) - z_i\,(3\gamma+5)\,(4\gamma+5)\,(\gamma+1) - x_i\,(\gamma+1)^2}{(3\gamma+5)\,(4\gamma+5)} \\
&= \frac{z_i\,(3\gamma+5)\,(4\gamma+5) + x_j\,(5\gamma+8)\,(\gamma+1) - x_i\,(\gamma+1)^2}{(3\gamma+5)\,(4\gamma+5)} \\
&= z_i + (\gamma+1)\,\frac{x_j\,(5\gamma+8) - x_i\,(\gamma+1)}{(3\gamma+5)\,(4\gamma+5)} \\
&= z_i + x_i\frac{(4\gamma+7)\,(\gamma+1)}{(3\gamma+5)\,(4\gamma+5)}.
\end{aligned}$$

That is,

$$s_{iz} = z_i + x_i\frac{(4\gamma+7)\,(\gamma+1)}{(3\gamma+5)\,(4\gamma+5)}.$$

Production subsidy granted to the export by the exporting firm:

$$
\begin{aligned}
& H\left(2y_i + s_{ix}\right) \\
= & \begin{vmatrix}
3(\gamma+2) & 3D_i(\gamma+2)(\gamma+1) + 3x_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) & \left(28\gamma+9\gamma^2+22\right) & 2(3\gamma+5)(\gamma+1) \\
-6(2\gamma+3) & -6D_i(2\gamma+3)(\gamma+1) + 3y_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) & -(3\gamma+5)(7\gamma+10) & 2(3\gamma+5)(\gamma+1) \\
(15\gamma+22) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & 2(2\gamma+3) & -2(2\gamma+3) \\
6(2\gamma+3) & 6D_i(2\gamma+3)(\gamma+1) - 3x_j(3\gamma+4)(5\gamma+8) - 3q_i^e(\gamma+1)(3\gamma+4) & -(9\gamma+14)(\gamma+1) & -(3\gamma+5)(7\gamma+10)
\end{vmatrix} \\
= & \begin{vmatrix}
4(3\gamma+5)(5\gamma+8) & -D_i(5\gamma+8)^2 + (3\gamma+4)(5\gamma+8)(3x_i + z_i(4\gamma+7)) & (5\gamma+8)^2 & -2(\gamma+2)(5\gamma+8) \\
(5\gamma+8)(12\gamma+17) & -D_i(5\gamma+8)(8\gamma+11) + (3\gamma+4)(5\gamma+8)(3y_i + z_i(4\gamma+7)) & -(5\gamma+8)(\gamma+1) & -2(\gamma+2)(5\gamma+8) \\
(15\gamma+22) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & 2(2\gamma+3) & -2(2\gamma+3) \\
(3\gamma+5)(5\gamma+8) & D_i(5\gamma+8)(\gamma+1) + (3\gamma+4)(5\gamma+8)\left(z_i(\gamma+1) - 3x_j\right) & -(5\gamma+8)(\gamma+1) & -(5\gamma+8)(5\gamma+7)
\end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix}
4(3\gamma+5) & -D_i(5\gamma+8) + (3\gamma+4)(3x_i + z_i(4\gamma+7)) & (5\gamma+8) & -2(\gamma+2) \\
(12\gamma+17) & -D_i(8\gamma+11) + (3\gamma+4)(3y_i + z_i(4\gamma+7)) & -(\gamma+1) & -2(\gamma+2) \\
(15\gamma+22) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & 2(2\gamma+3) & -2(2\gamma+3) \\
(3\gamma+5) & D_i(\gamma+1) + (3\gamma+4)\left(z_i(\gamma+1) - 3x_j\right) & -(\gamma+1) & -(5\gamma+7)
\end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix}
4(3\gamma+5) & (3\gamma+4)(3x_i + z_i(4\gamma+7)) & (5\gamma+8) & -2(\gamma+2) \\
(12\gamma+17) & -3D_i(3\gamma+4) + (3\gamma+4)(3y_i + z_i(4\gamma+7)) & -(\gamma+1) & -2(\gamma+2) \\
(15\gamma+22) & -D_i(3\gamma+4) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & 2(2\gamma+3) & -2(2\gamma+3) \\
(3\gamma+5) & (3\gamma+4)\left(z_i(\gamma+1) - 3x_j\right) & -(\gamma+1) & -(5\gamma+7)
\end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix}
4(3\gamma+5) & (3\gamma+4)(3x_i + z_i(4\gamma+7)) & (5\gamma+8) & -2(\gamma+2) \\
-3 & -3D_i(3\gamma+4) - 3(3\gamma+4)(x_i - y_i) & -3(2\gamma+3) & 0 \\
(15\gamma+22) & -D_i(3\gamma+4) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & 2(2\gamma+3) & -2(2\gamma+3) \\
(3\gamma+5) & (3\gamma+4)\left(z_i(\gamma+1) - 3x_j\right) & -(\gamma+1) & -(5\gamma+7)
\end{vmatrix} \\
= & -3(3\gamma+4)(5\gamma+8)^3 \begin{vmatrix}
4(3\gamma+5) & (3x_i + z_i(4\gamma+7)) & (5\gamma+8) & -2(\gamma+2) \\
1 & D_i + (x_i - y_i) & (2\gamma+3) & 0 \\
(15\gamma+22) & -D_i + z_i(5\gamma+8) + 3q_i^e & 2(2\gamma+3) & -2(2\gamma+3) \\
(3\gamma+5) & z_i(\gamma+1) - 3x_j & -(\gamma+1) & -(5\gamma+7)
\end{vmatrix} \\
= & -3(3\gamma+4)(5\gamma+8)^3 \begin{vmatrix}
4(3\gamma+5) & (3x_i + z_i(4\gamma+7)) & (5\gamma+8) & -2(\gamma+2) \\
1 & z_i + x_j + x_i & (2\gamma+3) & 0 \\
15\gamma+23 & z_i(5\gamma+8) + 2(2x_i + y_i) & 3(2\gamma+3) & -2(2\gamma+3) \\
(3\gamma+5) & z_i(\gamma+1) - 3x_j & -(\gamma+1) & -(5\gamma+7)
\end{vmatrix} \\
= & -3(3\gamma+4)(5\gamma+8)^3 \left(3x_i(3\gamma+4)\left(15\gamma+3\gamma^2+14\right) + 6y_i(3\gamma+4)(3\gamma+5)(4\gamma+5) + 3x_j(3\gamma+4)(9\gamma+13)(\gamma+1)\right) \\
= & -9(3\gamma+4)^2(5\gamma+8)^3 \left(x_i\left(15\gamma+3\gamma^2+14\right) + 2y_i(3\gamma+5)(4\gamma+5) + x_j(9\gamma+13)(\gamma+1)\right)
\end{aligned}
$$

$$
\begin{vmatrix}
3(\gamma+2) & 3D_i(\gamma+2)(\gamma+1) + 3x_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) & \left(28\gamma+9\gamma^2+22\right) & 2(3\gamma+5)(\gamma+1) \\
-6(2\gamma+3) & -6D_i(2\gamma+3)(\gamma+1) + 3y_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) & -(3\gamma+5)(7\gamma+10) & 2(3\gamma+5)(\gamma+1) \\
(15\gamma+22) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & 2(2\gamma+3) & -2(2\gamma+3) \\
6(2\gamma+3) & 6D_i(2\gamma+3)(\gamma+1) - 3x_j(3\gamma+4)(5\gamma+8) - 3q_i^e(\gamma+1)(3\gamma+4) & -(9\gamma+14)(\gamma+1) & -(3\gamma+5)(7\gamma+10)
\end{vmatrix}
$$

$$
\begin{aligned}
& 2y_i + s_{ix} \\
= & \frac{-9(3\gamma+4)(3\gamma+4)(5\gamma+8)^3\left(x_i\left(15\gamma+3\gamma^2+14\right) + 2y_i(3\gamma+5)(4\gamma+5) + x_j(9\gamma+13)(\gamma+1)\right)}{H} \\
= & \frac{x_i\left(15\gamma+3\gamma^2+14\right) + 2y_i(3\gamma+5)(4\gamma+5) + x_j(9\gamma+13)(\gamma+1)}{(3\gamma+5)(4\gamma+5)}
\end{aligned}
$$

$$
\begin{aligned}
s_{ix} &= \frac{x_i\left(15\gamma+3\gamma^2+14\right) + 2y_i(3\gamma+5)(4\gamma+5) + x_j(9\gamma+13)(\gamma+1)}{(3\gamma+5)(4\gamma+5)} - 2y_i \\
&= \frac{x_i\left(15\gamma+3\gamma^2+14\right) + x_j(9\gamma+13)(\gamma+1)}{(3\gamma+5)(4\gamma+5)}.
\end{aligned}
$$

That is,

$$s_{ix} = \frac{x_i\left(15\gamma + 3\gamma^2 + 14\right) + x_j\left(9\gamma + 13\right)\left(\gamma + 1\right)}{\left(3\gamma + 5\right)\left(4\gamma + 5\right)}.$$

Production subsidy granted to the domestic sales by the exporting firm:

$$\begin{aligned}
& H(2x_i + s_{iy}) \\
= & \begin{vmatrix} 3(\gamma+2) & -(3\gamma+5)(7\gamma+10) & 3D_i(\gamma+2)(\gamma+1) + 3x_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) & 2(3\gamma+5)(\gamma+1) \\ -6(2\gamma+3) & \left(28\gamma+9\gamma^2+22\right) & -6D_i(2\gamma+3)(\gamma+1) + 3y_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) & 2(3\gamma+5)(\gamma+1) \\ (15\gamma+22) & -(\gamma+2) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & -2(2\gamma+3) \\ 6(2\gamma+3) & 2(3\gamma+5)(\gamma+1) & 6D_i(2\gamma+3)(\gamma+1) - 3x_j(3\gamma+4)(5\gamma+8) - 3q_i^e(\gamma+1)(3\gamma+4) & -(3\gamma+5)(7\gamma+10) \end{vmatrix} \\
= & \begin{vmatrix} 4(3\gamma+5)(5\gamma+8) & -(5\gamma+8)^2 & -D_i(5\gamma+8)^2 + (3\gamma+4)(5\gamma+8)(3x_i + z_i(4\gamma+7)) & -2(\gamma+2)(5\gamma+8) \\ (5\gamma+8)(12\gamma+17) & (5\gamma+8)(\gamma+1) & -D_i(5\gamma+8)(8\gamma+11) + (3\gamma+4)(5\gamma+8)(3y_i + z_i(4\gamma+7)) & -2(\gamma+2)(5\gamma+8) \\ (15\gamma+22) & -(\gamma+2) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & -2(2\gamma+3) \\ (3\gamma+5)(5\gamma+8) & (5\gamma+8)(\gamma+1) & D_i(5\gamma+8)(\gamma+1) - \left(3x_j - z_i(\gamma+1)\right)(3\gamma+4)(5\gamma+8) & -(5\gamma+8)(5\gamma+7) \end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix} 4(3\gamma+5) & -(5\gamma+8) & -D_i(5\gamma+8) + (3\gamma+4)(3x_i + z_i(4\gamma+7)) & -2(\gamma+2) \\ (12\gamma+17) & (\gamma+1) & -D_i(8\gamma+11) + (3\gamma+4)(3y_i + z_i(4\gamma+7)) & -2(\gamma+2) \\ (15\gamma+22) & -(\gamma+2) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & -2(2\gamma+3) \\ (3\gamma+5) & (\gamma+1) & D_i(\gamma+1) - \left(3x_j - z_i(\gamma+1)\right)(3\gamma+4) & -(5\gamma+7) \end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix} 4(3\gamma+5) & -(5\gamma+8) & (3\gamma+4)(3x_i + z_i(4\gamma+7)) & -2(\gamma+2) \\ (12\gamma+17) & (\gamma+1) & -3D_i(3\gamma+4) + (3\gamma+4)(3y_i + z_i(4\gamma+7)) & -2(\gamma+2) \\ (15\gamma+22) & -(\gamma+2) & -2D_i(3\gamma+4) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) & -2(2\gamma+3) \\ (3\gamma+5) & (\gamma+1) & -\left(3x_j - z_i(\gamma+1)\right)(3\gamma+4) & -(5\gamma+7) \end{vmatrix} \\
= & (3\gamma+4)(5\gamma+8)^3 \begin{vmatrix} 4(3\gamma+5) & -(5\gamma+8) & 3x_i + z_i(4\gamma+7) & -2(\gamma+2) \\ (12\gamma+17) & (\gamma+1) & -3D_i + 3y_i + z_i(4\gamma+7) & -2(\gamma+2) \\ (15\gamma+22) & -(\gamma+2) & -2D_i + z_i(5\gamma+8) + 3q_i^e & -2(2\gamma+3) \\ (3\gamma+5) & (\gamma+1) & z_i(\gamma+1) - 3x_j & -(5\gamma+7) \end{vmatrix} \\
= & -3(3\gamma+4)(5\gamma+8)^3 \begin{vmatrix} 4(3\gamma+5) & -(5\gamma+8) & 3x_i + z_i(4\gamma+7) & -2(\gamma+2) \\ 1 & -(2\gamma+3) & D_i - y_i + x_i & 0 \\ (15\gamma+22) & -(\gamma+2) & -2D_i + z_i(5\gamma+8) + 3q_i^e & -2(2\gamma+3) \\ (3\gamma+5) & (\gamma+1) & z_i(\gamma+1) - 3x_j & -(5\gamma+7) \end{vmatrix} \\
= & -3(3\gamma+4)(5\gamma+8)^3 \begin{vmatrix} 4(3\gamma+5) & -(5\gamma+8) & 3x_i + z_i(4\gamma+7) & -2(\gamma+2) \\ 1 & -(2\gamma+3) & x_i + x_j + z_i & 0 \\ (15\gamma+22) & -(\gamma+2) & 3x_i + y_i - 2x_j + z_i(5\gamma+6) & -2(2\gamma+3) \\ (3\gamma+5) & (\gamma+1) & z_i(\gamma+1) - 3x_j & -(5\gamma+7) \end{vmatrix} \\
= & -3(3\gamma+4)(5\gamma+8)^3 \left(3(4\gamma+5)(3\gamma+5)(3\gamma+4)y_i + 9(\gamma+1)(5\gamma+8)(3\gamma+4)x_j + 3\left(64\gamma+21\gamma^2+47\right)(3\gamma+4)x_i\right) \\
= & -9(3\gamma+4)^2(5\gamma+8)^3\left((4\gamma+5)(3\gamma+5)y_i + 3(\gamma+1)(5\gamma+8)x_j + \left(64\gamma+21\gamma^2+47\right)x_i\right)
\end{aligned}$$

$$\begin{aligned}
2x_i + s_{iy} &= \frac{-9(3\gamma+4)^2(5\gamma+8)^3\left((4\gamma+5)(3\gamma+5)y_i + 3(\gamma+1)(5\gamma+8)x_j + \left(64\gamma+21\gamma^2+47\right)x_i\right)}{H} \\
&= \frac{-9(3\gamma+4)^2(5\gamma+8)^3\left((4\gamma+5)(3\gamma+5)y_i + 3(\gamma+1)(5\gamma+8)x_j + \left(64\gamma+21\gamma^2+47\right)x_i\right)}{-9(3\gamma+5)(4\gamma+5)(3\gamma+4)^2(5\gamma+8)^3} \\
&= \frac{(4\gamma+5)(3\gamma+5)y_i + 3(\gamma+1)(5\gamma+8)x_j + \left(64\gamma+21\gamma^2+47\right)x_i}{(4\gamma+5)(3\gamma+5)} \\
&= y_i + \frac{3(\gamma+1)(5\gamma+8)x_j + \left(64\gamma+21\gamma^2+47\right)x_i}{(4\gamma+5)(3\gamma+5)}
\end{aligned}$$

$$
\begin{aligned}
s_{iy} &= y_i + \frac{3(\gamma+1)(5\gamma+8)x_j + \left(64\gamma+21\gamma^2+47\right)x_i}{(4\gamma+5)(3\gamma+5)} - 2x_i \\
&= y_i + \frac{3(\gamma+1)(5\gamma+8)x_j + \left(64\gamma+21\gamma^2+47\right)x_i - 2(4\gamma+5)(3\gamma+5)x_i}{(4\gamma+5)(3\gamma+5)} \\
&= y_i + \frac{3(\gamma+1)(5\gamma+8)x_j - 3(\gamma+1)^2 x_i}{(4\gamma+5)(3\gamma+5)} \\
&= y_i + \frac{3(\gamma+1)((5\gamma+8)x_j - (\gamma+1)x_i)}{(4\gamma+5)(3\gamma+5)}.
\end{aligned}
$$

That is,

$$
s_{iy} = y_i + \frac{3(\gamma+1)((5\gamma+8)x_j - (\gamma+1)x_i)}{(4\gamma+5)(3\gamma+5)}.
$$

Import tariff imposed on the export by the foreign exporting firm:

$$
\begin{aligned}
& Ht_i \\
= & \begin{vmatrix}
3(\gamma+2) & -(3\gamma+5)(7\gamma+10) & \left(28\gamma+9\gamma^2+22\right) & 3D_i(\gamma+2)(\gamma+1) + 3x_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) \\
-6(2\gamma+3) & \left(28\gamma+9\gamma^2+22\right) & -(3\gamma+5)(7\gamma+10) & -6D_i(2\gamma+3)(\gamma+1) + 3y_i(3\gamma+4)(5\gamma+8) - 3q_i^e(4\gamma+7)(3\gamma+4) \\
(15\gamma+22) & -(\gamma+2) & 2(2\gamma+3) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) \\
6(2\gamma+3) & 2(3\gamma+5)(\gamma+1) & -(9\gamma+14)(\gamma+1) & 6D_i(2\gamma+3)(\gamma+1) - 3x_j(3\gamma+4)(5\gamma+8) - 3q_i^e(\gamma+1)(3\gamma+4)
\end{vmatrix} \\
= & \begin{vmatrix}
4(3\gamma+5)(5\gamma+8) & -(5\gamma+8)^2 & (5\gamma+8)^2 & -D_i(5\gamma+8)^2 + (3\gamma+4)(5\gamma+8)(3x_i + z_i(4\gamma+7)) \\
(5\gamma+8)(12\gamma+17) & (5\gamma+8)(\gamma+1) & -(5\gamma+8)(\gamma+1) & -D_i(5\gamma+8)(8\gamma+11) + (z_i(4\gamma+7)+3y_i)(3\gamma+4)(5\gamma+8) \\
(15\gamma+22) & -(\gamma+2) & 2(2\gamma+3) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) \\
(3\gamma+5)(5\gamma+8) & (5\gamma+8)(\gamma+1) & -(5\gamma+8)(\gamma+1) & D_i(5\gamma+8)(\gamma+1) + \left(z_i(3\gamma+4)(5\gamma+8)(\gamma+1) - 3x_j\right)(3\gamma+4)(5\gamma+8)
\end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix}
4(3\gamma+5) & -(5\gamma+8) & (5\gamma+8) & -D_i(5\gamma+8) + (3\gamma+4)(3x_i + z_i(4\gamma+7)) \\
(12\gamma+17) & (\gamma+1) & -(\gamma+1) & -D_i(8\gamma+11) + (z_i(4\gamma+7)+3y_i)(3\gamma+4) \\
(15\gamma+22) & -(\gamma+2) & 2(2\gamma+3) & -D_i(7\gamma+10) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) \\
(3\gamma+5) & (\gamma+1) & -(\gamma+1) & D_i(\gamma+1) + \left(z_i(\gamma+1) - 3x_j\right)(3\gamma+4)
\end{vmatrix} \\
= & (5\gamma+8)^3 \begin{vmatrix}
4(3\gamma+5) & 0 & (5\gamma+8) & (3\gamma+4)(3x_i + z_i(4\gamma+7)) \\
(12\gamma+17) & 0 & -(\gamma+1) & -3D_i(3\gamma+4) + (z_i(4\gamma+7)+3y_i)(3\gamma+4) \\
(15\gamma+22) & (3\gamma+4) & 2(2\gamma+3) & -D_i(3\gamma+4) + z_i(3\gamma+4)(5\gamma+8) + 3q_i^e(3\gamma+4) \\
(3\gamma+5) & 0 & -(\gamma+1) & \left(z_i(\gamma+1) - 3x_j\right)(3\gamma+4)
\end{vmatrix} \\
= & (5\gamma+8)^3(3\gamma+4)^2 \begin{vmatrix}
4(3\gamma+5) & 0 & (5\gamma+8) & 3x_i + z_i(4\gamma+7) \\
(12\gamma+17) & 0 & -(\gamma+1) & -3D_i + z_i(4\gamma+7) + 3y_i \\
(15\gamma+22) & 1 & 2(2\gamma+3) & -D_i + z_i(5\gamma+8) + 3q_i^e \\
(3\gamma+5) & 0 & -(\gamma+1) & z_i(\gamma+1) - 3x_j
\end{vmatrix} \\
= & (5\gamma+8)^3(3\gamma+4)^2 \begin{vmatrix}
4(3\gamma+5) & 0 & (5\gamma+8) & 3x_i + z_i(4\gamma+7) \\
(12\gamma+17) & 0 & -(\gamma+1) & 4z_i(\gamma+1) - 3x_j \\
(15\gamma+22) & 1 & 2(2\gamma+3) & 2\left(x_j+y_i\right) + z_i(5\gamma+7) \\
(3\gamma+5) & 0 & -(\gamma+1) & z_i(\gamma+1) - 3x_j
\end{vmatrix} \\
= & (5\gamma+8)^3(3\gamma+4)^2\left(9(\gamma+1)(3\gamma+4)x_i - 9(3\gamma+4)(5\gamma+8)x_j\right) \\
= & 9(5\gamma+8)^3(3\gamma+4)^3\left((\gamma+1)x_i - (5\gamma+8)x_j\right)
\end{aligned}
$$

Solve this for t_i to obtain

$$\begin{aligned} t_i &= \frac{9\left(5\gamma+8\right)^3\left(3\gamma+4\right)^3\left(\left(\gamma+1\right)x_i-\left(5\gamma+8\right)x_j\right)}{H} \\ &= \frac{9\left(5\gamma+8\right)^3\left(3\gamma+4\right)^3\left(\left(\gamma+1\right)x_i-\left(5\gamma+8\right)x_j\right)}{-9\left(3\gamma+5\right)\left(4\gamma+5\right)\left(3\gamma+4\right)^2\left(5\gamma+8\right)^3} \\ &= \frac{\left(3\gamma+4\right)\left(\left(5\gamma+8\right)x_j-\left(\gamma+1\right)x_i\right)}{\left(4\gamma+5\right)\left(3\gamma+5\right)}. \end{aligned}$$

That is,

$$t_i = \frac{\left(3\gamma+4\right)\left(\left(5\gamma+8\right)x_j-\left(\gamma+1\right)x_i\right)}{\left(4\gamma+5\right)\left(3\gamma+5\right)}.$$

We derived the following eight equilibrium conditions:

$$\begin{aligned} t_i &= \frac{\left(3\gamma+4\right)\left(\left(5\gamma+8\right)x_j-\left(\gamma+1\right)x_i\right)}{\left(4\gamma+5\right)\left(3\gamma+5\right)} \\ s_{ix} &= \frac{x_i\left(15\gamma+3\gamma^2+14\right)+x_j\left(9\gamma+13\right)\left(\gamma+1\right)}{\left(3\gamma+5\right)\left(4\gamma+5\right)} \\ s_{iy} &= y_i+\frac{3\left(\gamma+1\right)\left(\left(5\gamma+8\right)x_j-\left(\gamma+1\right)x_i\right)}{\left(4\gamma+5\right)\left(3\gamma+5\right)} \\ s_{iz} &= z_i+x_i\frac{\left(4\gamma+7\right)\left(\gamma+1\right)}{\left(3\gamma+5\right)\left(4\gamma+5\right)} \\ t_j &= \frac{\left(3\gamma+4\right)\left(\left(5\gamma+8\right)x_i-\left(\gamma+1\right)x_j\right)}{\left(4\gamma+5\right)\left(3\gamma+5\right)} \\ s_{jx} &= \frac{x_j\left(15\gamma+3\gamma^2+14\right)+x_i\left(9\gamma+13\right)\left(\gamma+1\right)}{\left(3\gamma+5\right)\left(4\gamma+5\right)} \\ s_{jy} &= y_j+\frac{3\left(\gamma+1\right)\left(\left(5\gamma+8\right)x_i-\left(\gamma+1\right)x_j\right)}{\left(4\gamma+5\right)\left(3\gamma+5\right)} \\ s_{jz} &= z_j+x_j\frac{\left(4\gamma+7\right)\left(\gamma+1\right)}{\left(3\gamma+5\right)\left(4\gamma+5\right)}, \end{aligned}$$

where all the outputs are expressed as the function of t_i, t_j, s_{ix}, s_{jx}, s_{iy}, s_{jy}, s_{iz}, s_{jz}.

$$
\begin{aligned}
x_i &= \frac{\left[\begin{array}{c} 3a\,(3\gamma+4)\,(\gamma+1) - 2\,(t_i - s_{jx})\,(3\gamma+5)\,(\gamma+1) - (t_j - s_{ix})\,(3\gamma+5)\,(7\gamma+10) \\ -s_{iy}\left(28\gamma+9\gamma^2+22\right) - s_{jy}\,(9\gamma+14)\,(\gamma+1) + 3s_{iz}\,(\gamma+2) - 6s_{jz}\,(2\gamma+3) \end{array}\right]}{3\,(3\gamma+4)\,(5\gamma+8)} \\
x_j &= \frac{\left[\begin{array}{c} 3a\,(3\gamma+4)\,(\gamma+1) - (t_i - s_{jx})\,(3\gamma+5)\,(7\gamma+10) - 2\,(t_j - s_{ix})\,(3\gamma+5)\,(\gamma+1) \\ -s_{iy}\,(9\gamma+14)\,(\gamma+1) - s_{jy}\left(28\gamma+9\gamma^2+22\right) - 6s_{iz}\,(2\gamma+3) + 3s_{jz}\,(\gamma+2) \end{array}\right]}{3\,(3\gamma+4)\,(5\gamma+8)} \\
y_i &= \frac{\left[\begin{array}{c} 3a\,(3\gamma+4)\,(\gamma+1) + (t_i - s_{jx})\,(9\gamma+14)\,(\gamma+1) + (t_j - s_{ix})\left(28\gamma+9\gamma^2+22\right) \\ +s_{iy}\,(3\gamma+5)\,(7\gamma+10) + 2s_{jy}\,(3\gamma+5)\,(\gamma+1) - 6s_{iz}\,(2\gamma+3) + 3s_{jz}\,(\gamma+2) \end{array}\right]}{3\,(3\gamma+4)\,(5\gamma+8)} \\
y_j &= \frac{\left[\begin{array}{c} 3a\,(3\gamma+4)\,(\gamma+1) + (t_i - s_{jx})\left(28\gamma+9\gamma^2+22\right) + (t_j - s_{ix})\,(9\gamma+14)\,(\gamma+1) \\ +2s_{iy}\,(3\gamma+5)\,(\gamma+1) + s_{jy}\,(3\gamma+5)\,(7\gamma+10) + 3s_{iz}\,(\gamma+2) - 6s_{jz}\,(2\gamma+3) \end{array}\right]}{3\,(3\gamma+4)\,(5\gamma+8)} \\
z_i &= \frac{3a\,(3\gamma+4) + 2\,(t_i - s_{jx})\,(2\gamma+3) - (t_j - s_{ix})\,(\gamma+2) - 2s_{iy}\,(2\gamma+3) + s_{jy}\,(\gamma+2) + s_{iz}\,(15\gamma+22) - 2s_{jz}}{(3\gamma+4)\,(5\gamma+8)} \\
z_j &= \frac{3a\,(3\gamma+4) - (t_i - s_{jx})\,(\gamma+2) + 2\,(t_j - s_{ix})\,(2\gamma+3) + s_{iy}\,(\gamma+2) - 2s_{jy}\,(2\gamma+3) - 2s_{iz} + s_{jz}\,(15\gamma+22)}{(3\gamma+4)\,(5\gamma+8)}
\end{aligned}
$$

$$
t_i = \frac{(3\gamma+4)\left[\begin{array}{l} (5\gamma+8)\,\frac{3a(3\gamma+4)(\gamma+1)-(t_i-s_{jx})(3\gamma+5)(7\gamma+10)-2(t_j-s_{ix})(3\gamma+5)(\gamma+1)-s_{iy}(9\gamma+14)(\gamma+1)-s_{jy}\left(28\gamma+9\gamma^2+22\right)-6s_{iz}(2\gamma+3)+3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)} \\ -(\gamma+1)\,\frac{3a(3\gamma+4)(\gamma+1)-2\left(t_i-s_{jx}\right)(3\gamma+5)(\gamma+1)-\left(t_j-s_{ix}\right)(3\gamma+5)(7\gamma+10)-s_{iy}\left(28\gamma+9\gamma^2+22\right)-s_{jy}(9\gamma+14)(\gamma+1)+3s_{iz}(\gamma+2)-6s_{jz}(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)} \end{array}\right]}{(4\gamma+5)(3\gamma+5)},
$$

$$
s_{ix} = \frac{\left[\begin{array}{l} \frac{3a(3\gamma+4)(\gamma+1)-2\left(t_i-s_{jx}\right)(3\gamma+5)(\gamma+1)-\left(t_j-s_{ix}\right)(3\gamma+5)(7\gamma+10)-s_{iy}\left(28\gamma+9\gamma^2+22\right)-s_{jy}(9\gamma+14)(\gamma+1)+3s_{iz}(\gamma+2)-6s_{jz}(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)}\,(15\gamma+3\gamma^2+14) \\ +\frac{3a(3\gamma+4)(\gamma+1)-\left(t_i-s_{jx}\right)(3\gamma+5)(7\gamma+10)-2\left(t_j-s_{ix}\right)(3\gamma+5)(\gamma+1)-s_{iy}(9\gamma+14)(\gamma+1)-s_{jy}\left(28\gamma+9\gamma^2+22\right)-6s_{iz}(2\gamma+3)+3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)}\,(9\gamma+13)\,(\gamma+1) \end{array}\right]}{(3\gamma+5)(4\gamma+5)},
$$

$$
\begin{aligned}
s_{iy} = {} & \frac{3a(3\gamma+4)(\gamma+1) + (t_i - s_{jx})(9\gamma+14)(\gamma+1) + (t_j - s_{ix})(28\gamma+9\gamma^2+22)}{3(3\gamma+4)(5\gamma+8)} \\
& + \frac{s_{iy}(3\gamma+5)(7\gamma+10) + 2s_{jy}(3\gamma+5)(\gamma+1) - 6s_{iz}(2\gamma+3) + 3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)} \\
& + \frac{3(\gamma+1)\left[\begin{array}{l} (5\gamma+8)\,\frac{3a(3\gamma+4)(\gamma+1)-(t_i-s_{jx})(3\gamma+5)(7\gamma+10)-2(t_j-s_{ix})(3\gamma+5)(\gamma+1)-s_{iy}(9\gamma+14)(\gamma+1)-s_{jy}(28\gamma+9\gamma^2+22)-6s_{iz}(2\gamma+3)+3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)} \\ -(\gamma+1)\,\frac{3a(3\gamma+4)(\gamma+1)-2(t_i-s_{jx})(3\gamma+5)(\gamma+1)-(t_j-s_{ix})(3\gamma+5)(7\gamma+10)-s_{iy}(28\gamma+9\gamma^2+22)-s_{jy}(9\gamma+14)(\gamma+1)+3s_{iz}(\gamma+2)-6s_{jz}(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)} \end{array}\right]}{(4\gamma+5)(3\gamma+5)},
\end{aligned}
$$

$$
\begin{aligned}
s_{iz} = {} & \frac{3a\,(3\gamma+4) + 2(t_i - s_{jx})\,(2\gamma+3) - (t_j - s_{ix})\,(\gamma+2) - 2s_{iy}\,(2\gamma+3) + s_{jy}\,(\gamma+2) + s_{iz}\,(15\gamma+22) - 2s_{jz}}{(3\gamma+4)\,(5\gamma+8)} \\
& + \frac{3a\,(3\gamma+4)\,(\gamma+1) - 2\,(t_i - s_{jx})\,(3\gamma+5)\,(\gamma+1) - (t_j - s_{ix})\,(3\gamma+5)\,(7\gamma+10) - s_{iy}\left(28\gamma+9\gamma^2+22\right)}{(3\gamma+5)\,(4\gamma+5)} \\
& - \frac{s_{jy}\,(9\gamma+14)\,(\gamma+1) + 3s_{iz}\,(\gamma+2) - 6s_{jz}\,(2\gamma+3)}{3\,(3\gamma+4)\,(5\gamma+8)}\,\frac{(4\gamma+7)\,(\gamma+1)}{(3\gamma+5)\,(4\gamma+5)},
\end{aligned}
$$

$$
t_j = \frac{(3\gamma+4)\left[\begin{array}{l} (5\gamma+8)\,\frac{3a(3\gamma+4)(\gamma+1)-2\left(t_i-s_{jx}\right)(3\gamma+5)(\gamma+1)-\left(t_j-s_{ix}\right)(3\gamma+5)(7\gamma+10)-s_{iy}\left(28\gamma+9\gamma^2+22\right)-s_{jy}(9\gamma+14)(\gamma+1)+3s_{iz}(\gamma+2)-6s_{jz}(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)} \\ -(\gamma+1)\,\frac{3a(3\gamma+4)(\gamma+1)-\left(t_i-s_{jx}\right)(3\gamma+5)(7\gamma+10)-2\left(t_j-s_{ix}\right)(3\gamma+5)(\gamma+1)-s_{iy}(9\gamma+14)(\gamma+1)-s_{jy}\left(28\gamma+9\gamma^2+22\right)-6s_{iz}(2\gamma+3)+3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)} \end{array}\right]}{(4\gamma+5)(3\gamma+5)},
$$

$$
\begin{aligned}
s_{jx} = {} & \frac{\frac{3a(3\gamma+4)(\gamma+1)-\left(t_i-s_{jx}\right)(3\gamma+5)(7\gamma+10)-2\left(t_j-s_{ix}\right)(3\gamma+5)(\gamma+1)-s_{iy}(9\gamma+14)(\gamma+1)-s_{jy}\left(28\gamma+9\gamma^2+22\right)-6s_{iz}(2\gamma+3)+3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)}\,(15\gamma+3\gamma^2+14)}{(3\gamma+5)(4\gamma+5)} \\
& + \frac{\frac{3a(3\gamma+4)(\gamma+1)-2(t_i-s_{jx})(3\gamma+5)(\gamma+1)-\left(t_j-s_{ix}\right)(3\gamma+5)(7\gamma+10)-s_{iy}\left(28\gamma+9\gamma^2+22\right)-s_{jy}(9\gamma+14)(\gamma+1)+3s_{iz}(\gamma+2)-6s_{jz}(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)}\,(9\gamma+13)\,(\gamma+1)}{(3\gamma+5)(4\gamma+5)},
\end{aligned}
$$

$$
\begin{aligned}
s_{jy} = & \frac{3a(3\gamma+4)(\gamma+1) + (t_i - s_{jx})(28\gamma+9\gamma^2+22) + (t_j - s_{ix})(9\gamma+14)(\gamma+1) + 2s_{iy}(3\gamma+5)(\gamma+1) + s_{jy}(3\gamma+5)(7\gamma+10)}{3(3\gamma+4)(5\gamma+8)} \\
& + \frac{3s_{iz}(\gamma+2) - 6s_{jz}(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)} \\
& + \frac{3(\gamma+1)\left[\begin{array}{l} (5\gamma+8)\frac{3a(3\gamma+4)(\gamma+1)-2(t_i-s_{jx})(3\gamma+5)(\gamma+1)-(t_j-s_{ix})(3\gamma+5)(7\gamma+10)-s_{iy}(28\gamma+9\gamma^2+22)-s_{jy}(9\gamma+14)(\gamma+1)+3s_{iz}(\gamma+2)-6s_{jz}(2\gamma+3)}{3(3\gamma+4)(5\gamma+8)} \\ -(\gamma+1)\frac{3a(3\gamma+4)(\gamma+1)-(t_i-s_{jx})(3\gamma+5)(7\gamma+10)-2(t_j-s_{ix})(3\gamma+5)(\gamma+1)-s_{iy}(9\gamma+14)(\gamma+1)-s_{jy}(28\gamma+9\gamma^2+22)-6s_{iz}(2\gamma+3)+3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)} \end{array}\right]}{(4\gamma+5)(3\gamma+5)}, \\
s_{jz} = & \frac{3a(3\gamma+4) - (t_i - s_{jx})(\gamma+2) + 2(t_j - s_{ix})(2\gamma+3) + s_{iy}(\gamma+2) - 2s_{jy}(2\gamma+3) - 2s_{iz} + s_{jz}(15\gamma+22)}{(3\gamma+4)(5\gamma+8)} \\
& + \frac{3a(3\gamma+4)(\gamma+1) - (t_i - s_{jx})(3\gamma+5)(7\gamma+10) - 2(t_j - s_{ix})(3\gamma+5)(\gamma+1)}{(3\gamma+4)(5\gamma+8)} \\
& - \frac{s_{iy}(9\gamma+14)(\gamma+1) - s_{jy}(28\gamma+9\gamma^2+22) - 6s_{iz}(2\gamma+3) + 3s_{jz}(\gamma+2)}{3(3\gamma+4)(5\gamma+8)} \frac{(4\gamma+7)(\gamma+1)}{(3\gamma+5)(4\gamma+5)}.
\end{aligned}
$$

$$
t_i = t_j = 0; \qquad s_{ix} = s_{jx} = 0; \qquad s_{iy} = s_{jy} = \frac{a\gamma}{2\gamma+1} > s_{iz} = s_{jz} = \frac{a}{2\gamma+1}.
$$

The Nash equilibrium outputs evaluated at the Nash equilibrium policy instruments are

$$
x_i = x_j = 0; \qquad y_i = y_j = \frac{a\gamma}{2\gamma+1} > z_i = z_j = \frac{a}{2\gamma+1}
$$

with the following proof:

$$
\begin{aligned}
x_i = & \frac{3a\left(3\gamma+4\right)\left(\gamma+1\right) - 2\left(t_i - s_{jx}\right)\left(3\gamma+5\right)\left(\gamma+1\right) - \left(t_j - s_{ix}\right)\left(3\gamma+5\right)\left(7\gamma+10\right) - s_{iy}\left(28\gamma+9\gamma^2+22\right)}{3(3\gamma+4)(5\gamma+8)} \\
& - \frac{s_{jy}\left(9\gamma+14\right)\left(\gamma+1\right) + 3s_{iz}\left(\gamma+2\right) - 6s_{jz}\left(2\gamma+3\right)}{3(3\gamma+4)(5\gamma+8)} \Bigg|_{t_i=0, t_j=0, s_{ix}=0, s_{iy}=a\frac{\gamma}{2\gamma+1}, s_{jx}=0, s_{iz}=\frac{a}{2\gamma+1}, s_{jy}=a\frac{\gamma}{2\gamma+1}, s_{jz}=\frac{a}{2\gamma+1}}, \\
x_j = & \frac{3a\left(3\gamma+4\right)\left(\gamma+1\right) - \left(t_i - s_{jx}\right)\left(3\gamma+5\right)\left(7\gamma+10\right) - 2\left(t_j - s_{ix}\right)\left(3\gamma+5\right)\left(\gamma+1\right) - s_{iy}\left(9\gamma+14\right)\left(\gamma+1\right)}{3(3\gamma+4)(5\gamma+8)} \\
& - \frac{s_{jy}\left(28\gamma+9\gamma^2+22\right) - 6s_{iz}\left(2\gamma+3\right) + 3s_{jz}\left(\gamma+2\right)}{3(3\gamma+4)(5\gamma+8)} \Bigg|_{t_i=0, t_j=0, s_{ix}=0, s_{iy}=a\frac{\gamma}{2\gamma+1}, s_{jx}=0, s_{iz}=\frac{a}{2\gamma+1}, s_{jy}=a\frac{\gamma}{2\gamma+1}, s_{jz}=\frac{a}{2\gamma+1}}, \\
y_i = & \frac{3a\left(3\gamma+4\right)\left(\gamma+1\right) + \left(t_i - s_{jx}\right)\left(9\gamma+14\right)\left(\gamma+1\right) + \left(t_j - s_{ix}\right)\left(28\gamma+9\gamma^2+22\right) + s_{iy}\left(3\gamma+5\right)\left(7\gamma+10\right)}{3(3\gamma+4)(5\gamma+8)} \\
& + \frac{2s_{jy}\left(3\gamma+5\right)\left(\gamma+1\right) - 6s_{iz}\left(2\gamma+3\right) + 3s_{jz}\left(\gamma+2\right)}{3(3\gamma+4)(5\gamma+8)} \Bigg|_{t_i=0, t_j=0, s_{ix}=0, s_{iy}=a\frac{\gamma}{2\gamma+1}, s_{jx}=0, s_{iz}=\frac{a}{2\gamma+1}, s_{jy}=a\frac{\gamma}{2\gamma+1}, s_{jz}=\frac{a}{2\gamma+1}}, \\
y_j = & \frac{3a\left(3\gamma+4\right)\left(\gamma+1\right) + \left(t_i - s_{jx}\right)\left(28\gamma+9\gamma^2+22\right) + \left(t_j - s_{ix}\right)\left(9\gamma+14\right)\left(\gamma+1\right) + 2s_{iy}\left(3\gamma+5\right)\left(\gamma+1\right)}{3(3\gamma+4)(5\gamma+8)} \\
& + \frac{s_{jy}\left(3\gamma+5\right)\left(7\gamma+10\right) + 3s_{iz}\left(\gamma+2\right) - 6s_{jz}\left(2\gamma+3\right)}{3(3\gamma+4)(5\gamma+8)} \Bigg|_{t_i=0, t_j=0, s_{ix}=0, s_{iy}=a\frac{\gamma}{2\gamma+1}, s_{jx}=0, s_{iz}=\frac{a}{2\gamma+1}, s_{jy}=a\frac{\gamma}{2\gamma+1}, s_{jz}=\frac{a}{2\gamma+1}}, \\
z_i = & \frac{3a\left(3\gamma+4\right) + 2\left(t_i - s_{jx}\right)\left(2\gamma+3\right) - \left(t_j - s_{ix}\right)\left(\gamma+2\right) - 2s_{iy}\left(2\gamma+3\right)}{3(3\gamma+4)(5\gamma+8)} \\
& + \frac{s_{jy}\left(\gamma+2\right) + s_{iz}\left(15\gamma+22\right) - 2s_{jz}}{\left(3\gamma+4\right)\left(5\gamma+8\right)} \Bigg|_{t_i=0, t_j=0, s_{ix}=0, s_{iy}=a\frac{\gamma}{2\gamma+1}, s_{jx}=0, s_{iz}=\frac{a}{2\gamma+1}, s_{jy}=a\frac{\gamma}{2\gamma+1}, s_{jz}=\frac{a}{2\gamma+1}}, \\
z_j = & \frac{\left[\begin{array}{l} 3a(3\gamma+4) - (t_i - s_{jx})(\gamma+2) + 2(t_j - s_{ix})(2\gamma+3) \\ +s_{iy}(\gamma+2) - 2s_{jy}(2\gamma+3) - 2s_{iz} + s_{jz}(15\gamma+22) \end{array}\right]}{(3\gamma+4)(5\gamma+8)} \Bigg|_{t_i=0=t_j=s_{ix}=s_{jx}, s_{iy}=s_{jy}=\frac{a\gamma}{2\gamma+1}, s_{iz}=s_{jz}=\frac{a}{2\gamma+1}}.
\end{aligned}
$$

and the social welfare

$$\begin{aligned}
SW_i &\equiv CS_i + \pi_i^d + \pi_i^e - s_{ix}x_i - s_{iy}y_i - s_{iz}z_i + t_i x_j \\
&= \frac{1}{2}D_i^2 + \left(\frac{1}{2}\gamma + 1\right) z_i^2 + \frac{3}{2}\left(q_i^e\right)^2 - 2x_i y_i - s_{ix}x_i - s_{iy}y_i - s_{iz}z_i + t_i x_j \\
&= \frac{1}{2}\left(y_i + z_i + 0\right)^2 + \left(\frac{1}{2}\gamma + 1\right) z_i^2 + \frac{3}{2}\left(y_i + 0\right)^2 - 2 \times 0 \times y_i - 0 \times 0 - s_{iy}y_i - s_{iz}z_i + 0 \times 0 \\
&= \frac{1}{2}\left(\frac{a\gamma}{2\gamma+1} + \frac{a}{2\gamma+1}\right)^2 + \left(\frac{1}{2}\gamma+1\right)\left(\frac{a}{2\gamma+1}\right)^2 + \frac{3}{2}\left(\frac{a\gamma}{2\gamma+1}\right)^2 - \frac{a\gamma}{2\gamma+1}\frac{a\gamma}{2\gamma+1} - \frac{a}{2\gamma+1}\frac{a}{2\gamma+1} \\
&= \frac{a^2\left(\gamma+1\right)}{2\left(2\gamma+1\right)}
\end{aligned}$$

These imply Proposition 8. □

IV. Many Firms: m Non-Exporting and n Exporting Firms

$$\begin{aligned}
e_i^*(m,n,\gamma) &= -\frac{a\left[n\gamma N + \widehat{m}\left(\gamma+1\right)\left(m+\widehat{N}\right)\right]}{n\gamma^2 N + \widehat{m}\left[\widehat{m}(m+\widehat{N}) + \gamma\left(\gamma+1\right)NM\right]} \\
&\quad + \frac{cn\left[\left(N+\gamma\right)\left(m+\left(\gamma+1\right)N+\widehat{N}\right) - \left(\gamma+1\right)\left(n+\gamma+1+\widehat{m}(m+\widehat{N})\right)\right]}{\left(\gamma+1\right)\left\{n\gamma^2 N + \widehat{m}\left[\widehat{m}(m+\widehat{N}) + \gamma\left(\gamma+1\right)NM\right]\right\}}, \\
s_i^*(m,n,\gamma) &= \frac{a\beta\left(n\gamma N + \widehat{m}\left(\gamma+1\right)\left(m+\widehat{N}\right)\right) - cn\left(\left(m+\gamma\right)N + \left(\gamma+1\right)\left(m+\widehat{N}\right)\right)}{n\gamma^2 N + \widehat{m}\left[\widehat{m}(m+\widehat{N}) + \gamma\left(\gamma+1\right)NM\right]}, \\
t_i^*(m,n,\gamma) &= \frac{cn\left(N-1\right)\left(\widehat{N} - m\gamma\right)}{\left(\gamma+1\right)\left\{n\gamma^2 N + \widehat{m}\left[\widehat{m}(m+\widehat{N}) + \gamma\left(\gamma+1\right)NM\right]\right\}}.
\end{aligned}$$

The optimal policies (e_i^*, s_i^*, t_i^*) are related with the Nash equilibrium outputs $(q_{ii}^{e*}, q_{ij}^{e*}, q_i^{d*})$ in the following manner:

$$\begin{aligned}
e_i^* &= \frac{\left(n+\gamma+1\right)q_{ij}^{e*}}{N} - \frac{\left(m\left(\gamma+1\right)+n\gamma\right)q_{ii}^{e*} + nq_i^{d*}}{\left(\gamma+1\right)\widehat{m}}, \\
s_i^* &= \frac{mq_{ii}^{e*} + nq_i^{d*}}{\widehat{m}}, \qquad t_i^* = \frac{(\widehat{N} - m\gamma)q_{ij}^{e*}}{\beta N}.
\end{aligned}$$

Therefore, assuming interior solutions, we can show $s_i^* > 0$ and $t_i^* > 0$. To explore the possibility of $e_i^* > 0$, we write the condition for $e_i^* > 0$ as

$$a < E(m, n, \gamma)\, cn,$$

where

$$E(m, n, \gamma) \equiv \frac{(N+\gamma)\left[m + (\gamma+1)N + \widehat{N}\right] - (\gamma+1)\left[n+\gamma+1+\widehat{m}(m+\widehat{N})\right]}{(\gamma+1)\left[n\gamma N + \widehat{m}(\gamma+1)(m+\widehat{N})\right]}.$$

On the other hand, q_i^{d*} is explicitly derived as follows:

$$q_i^{d*} = \frac{a(\gamma+1)\widehat{m}\left[n\gamma N + \widehat{m}(\gamma+1)(m+\widehat{N})\right] - cn\left[(m+\gamma)\left(\widehat{m}(m+\widehat{N}) + nN\right) + n(m+\widehat{N})\right]}{n\left\{n\gamma^2 N + \widehat{m}[\widehat{m}(m+\widehat{N}) + \gamma(\gamma+1)NM]\right\}}.$$

Therefore, we write the condition for $q_i^{d*} > 0$ as

$$a > K(m, n, \gamma)\, cn,$$

where

$$K(m, n, \gamma) \equiv \frac{(m+\gamma)\left[\widehat{m}(m+\widehat{N}) + nN\right] + n(m+\widehat{N})}{\widehat{m}(\gamma+1)\left[n\gamma N + \widehat{m}(\gamma+1)(m+\widehat{N})\right]}.$$

If $K(m, n, \gamma) < E(m, n, \gamma)$ is met for some γ, m, and n, then there exist a and c such that the following condition is satisfied:

$$K(m, n, \gamma)\, cn < a < E(m, n, \gamma)\, cn.$$

This implies that both $q_i^{d*} > 0$ and $s_i^* > 0$ hold for such γ, m, n, a and c.

We thus examine the condition of $K(m, n, \gamma) < E(m, n, \gamma)$. First, we set $\gamma = 0$. From

$$\begin{aligned} K(m, n, 0) &= \frac{(n + m(m+n))(2m+n+1) + mn(m+n+1)}{(m+n)^2(2m+n+1)}, \\ E(m, n, 0) &= \frac{(m+n)^2 + 4m + 2n + 1}{(m+n)(2m+n+1)}, \end{aligned}$$

we calculate:

$$E(m,n,0)-K(m,n,0)=\frac{m(2m+1)+(n+1-m)(m+n)^2}{(m+n)^2(2m+n+1)}.$$

If $n+1>m$ holds, it is sufficient to ensure $E(m,n,0)>K(m,n,0)$ and thus we can generate the case for $e_i^*>0$. If the number of the exporting firms is relatively small, the export subsidy becomes optimal. This is because competition among domestic exporting firms is not so keen that the profit-shifting effect of export subsidy dominates the negative terms of trade effect.

We next set $\gamma=1$. Similarly, from

$$\begin{aligned} K(m,n,1) &= \frac{(2n+m(m+n+1))(4m+3n+6)+n(m+1)(m+n+2)}{2(m+n)(n(m+n+2)+2(m+n)(4m+3n+6))}, \\ E(m,n,1) &= \frac{16m+11n+26-(m+n)(2m+n)}{2(n(m+n+2)+2(m+n)(4m+3n+6))}, \end{aligned}$$

we calculate:

$$E(m,n,1)-K(m,n,1)=\frac{20m+3n(n+4)-(m+n)(m+n-1)(6m+n)}{2(m+n)(n(m+n+2)+2(m+n)(4m+3n+6))}.$$

We further set $m=n$ in the above expression to get $E(n,n,1)-K(n,n,1)=\frac{17n-28n^2+32}{8n(15n+13)}\equiv f(n)$. Accordingly, we have $f(1)=\frac{3}{32}>0$ and $f(2)=-\frac{23}{344}<0$. In the case of $\gamma=1$ and $m=n=2$, an export tax becomes optimal.

References

[1] Samuelson, P. A. (1983) *Foundations of Economic Analysis*, Enlarged edition, Harvard University Press.