Toward a Theory of the Labor Share's Fall: A Dynamic Model of the "Superstar" Firm

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Toward a Theory of the Labor Share's Fall: A Dynamic Model of the "Superstar" Firm

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Abstract In the last few years, there have been a growing number of studies pointing to a decline in the labor share, and this new fact has been clearly confirmed by Autor et al. and Karabarbounis and Neiman, among others. One of their key findings is that the decline in the labor share is primarily driven by reallocation among firms, rather than the decline in the weighted average labor share within firms (between-firm effect). On the other hand, at the industry level, the aggregate macro labor share is mainly affected by within-industry effects (the movement of the labor share in each industry). To explain these facts simultaneously, Autor et al. propose the "superstar" firm theory. That is, the aggregate labor share of an industry tends to decline as the most productive firms with the lowest labor share, called "superstar" firms, increasingly dominate the industry. Their theory is analyzed completely within the framework of statics, even though it clearly has many dynamical aspects. In this study, we propose a multiple-firm optimal growth model as a benchmark model that provides a solid theoretical foundation for the superstar firm theory. The multi-firm optimal growth model was intensively investigated by J. Scheinkman and L. McKenzie, under the title "Turnpike theory for multisector optimal growth model." The focus here is on the dynamic theory of superstar firms, but as the causes of the declining labor income share are as diverse as those reported in the McKinsey Discussion Paper, the policies that address them must also be diverse.
(251 words)

Keywords: between-firm effect, within-industry effect, structural change, multi-firm optimal growth model, turnpike theory, Euler equations, saddle-path stability, global stability, Neumann-McKenzie facet

JEL: O11, O31, O4

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1 Introduction

Over the past several years, we have witnessed an increasing number of studies that indicate the decline of the labor share. This phenomenon has been clearly confirmed by Autor et al. (2017). One of their important findings is that “a fall in the labor share will be driven largely by between-firm reallocation rather than a fall in the unweighted mean labor share within firms.” To explain this fact, they have presented the “superstar-firm” theory. Since the most productive firms with a low labor share become increasingly dominate the industry, the aggregated labor share will tend to fall. Although their theory clearly has many dynamical aspects, it is totally analyzed within the framework of statics. We need a dynamic model for a theoretical explanation of the phenomenon.

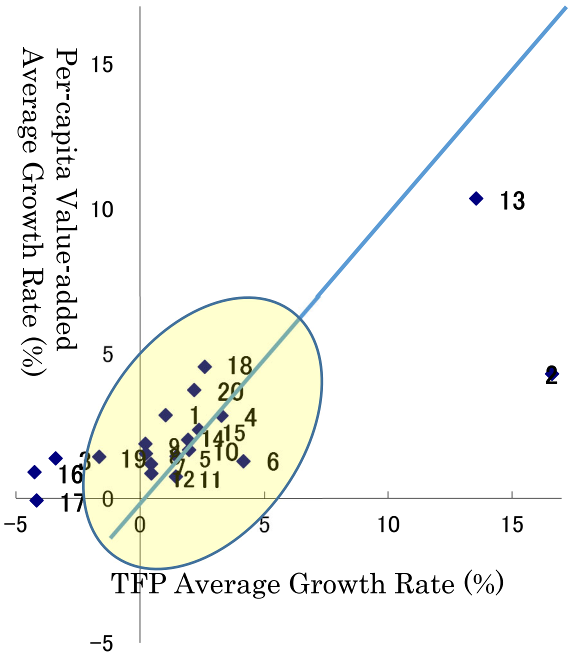
However, we cannot apply the macro-dynamic models used in many macro growth research, because they are mainly built for focusing on the “Kaldor facts” and firmly based on the Solow growth model. Since the core of the Kaldor facts is now empirically refuted, we need an alternative dynamic model to explain the new phenomenon.

I would like to propose here a multisectoral optimal growth model as a model that provides a theoretical basis for "superstar" firms. Here, we will read sector as firm. The multisectoral optimal growth model was intensively examined by J. Scheinkman, L. McKenzie and others in the 1970s and the 1980s under the title of the turnpike theory. Furthermore, based on the turnpike theory, Takahashi (1985) demonstrated the existence of an optimal stationary path and global stability in a multi-sector optimal growth model with one pure consumption goods sector and n consumable capital goods sectors. In the following, I will briefly explain how this approach can be applied to the superstar firm theory.

The chapter is organized as follows. In Section 4.2, we will examine some empirical facts and present two important facts that should be demonstrated theoretically. In Section 4.3, a multi-firm optimal growth model with firm specific TFP will be presented. In Section 4.4, the overtime optimization problem will be rewritten into the reduced form optimization problem. In Section 4.5 the reduced form optimization problem will be solved by applying turnpike theory. In Section 4.6, the results obtained in Section 4.5 will be used to illustrate two important observations reported in Section 4.2 and we will conclude the paper and provide some policy implications. Appendix A includes vector and matrix notation. In Appendix B, to help in understanding the important equations derived in Section 4.3.2, we consider two-sector model and confirm our results.

2 Some empirical facts

The past few years have witnessed many efforts to collect and archive the industry-level databases among countries. One of these databases is now easily accessed on the Web: the Eu-Klems Growth and Productivity Database. It covers 31 countries and 78 industries from 1970 and contains each industry's value-added and the TFP growth rate. Fig.1 shows the relationship between the average per capita growth rate of the value-added and the average industry-level TFP growth rate at the US industry level from 1970 to 2005. Fig.2 presents the same data for the Japanese economy. If an industry were on the 45-degree line, this would imply that the industry's per capita value-added would grow at its TFP growth rate. We can conclude from the figures that in both countries, the majority of industries cluster around the 45-degree line.



#	INDUSTRY
1	TOTAL INDUSTRIES
2	AGRICULTURE, HUNTING, FORESTRY AND FISHING
3	MINING AND QUARRYING
4	TOTAL MANUFACTURING
5	FOOD , BEVERAGES AND TOBACCO
6	TEXTILES, TEXTILE , LEATHER AND FOOTWEAR
7	WOOD AND OF WOOD AND CORK
8	PULP, PAPER, PAPER , PRINTING AND PUBLISHING
9	CHEMICAL, RUBBER, PLASTICS AND FUEL
10	OTHER NON-METALLIC MINERAL
11	BASIC METALS AND FABRICATED METAL
12	MACHINERY, NEC
13	ELECTRICAL AND OPTICAL EQUIPMENT
14	TRANSPORT EQUIPMENT
15	MANUFACTURING NEC; RECYCLING
16	ELECTRICITY, GAS AND WATER SUPPLY
17	CONSTRUCTION
18	WHOLESALE AND RETAIL TRADE
19	HOTELS AND RESTAURANTS
20	TRANSPORT AND STORAGE AND COMMUNICATION

Fig.1 U.S. Economy (1970 – 2005). Source: EU-KLEMS Database

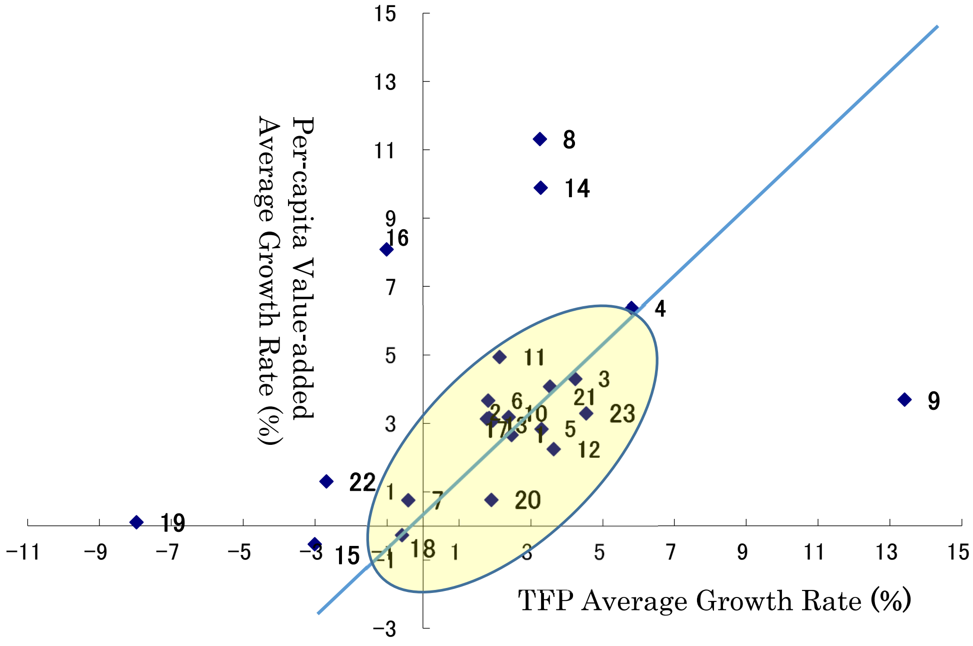


Fig.2 Japanese Economy (1970 – 2005). Source: EU-KLEMS Database

We can summarize these facts as follows.

Fact 1 (**Structural change**): Each industry has its own steady state with an industry-specific growth rate, which is highly correlated to its own TFP growth rate.

Following Karabarbounis and Neiman (2015), let us introduce the standard “within-between” accounting decomposition of the aggregated labor shares (s_L) as follows:

$$(*)\quad \Delta s_L = \underbrace{\sum_i \overline{\omega}_i \Delta s_{Li}}_{\textbf{Within-effect term}} + \underbrace{\sum_i \overline{s}_{Li} \Delta \omega_i}_{\textbf{Between-effect term}}$$

where the index i stands for firms or industries, ω_i denotes the ith labor share in total value added, $\overline{x}$ denotes the arithmatic mean of the variable x, and Δx denotes the estimated linear trend in x.

OECD (2003) examined the industry-level productivity growth in detail and reported the following observation:

Fact 2 (**Within-industry effect**): A large contribution to the overall productivity growth patterns comes from productivity changes within industries, rather than the significant shifts in employment across industries.

Furthermore, Karabarbounis and Neiman (2015) did industry-based empirical research and reported that more than 90% of the labor share decline reflects within-industry effect: Each industry’s labor share changes due to the within-effect denoted by the first term of (*), but not due to the value-added changes denoted by the second term of (*). This fact also reconfirms Fact 2.

Autor et al. (2017), on the other hand, examined the firm-based decomposition of industry's labor share movements and concluded that the reallocation among incumbents ("between-firm effect") was the main components of the fall of the labor share. They also reported that the reallocation component among incumbents was three times higher than the within-firm component from 1982 to 1997 and seven times higher from 1997 to 2012. They further stated that since the most productive firms with a low share of labor become increasingly dominate an industry, this industry's labor share will tend to fall. They call such a firm a "superstar-firm". We summarize this as follows.

Fact 3 (**Between-firm effect**): Due to the "super-star" firms, an industry's labor share declines.

In summary, we need to develop a theoretical model that simultaneously explains the following three empirical facts: structural changes, between-firm effects, and within-industry effects. These new stylized facts cannot be explained by standard macro growth models. Therefore, it is necessary to construct a firm-based multi-sector model as shown in the following sections.

3 The model and assumptions

We begin with constructing a structural model, and we will set up the neoclassical multisector optimal growth model with a sector specific TFP term (in other words, the sector specific Hicks-neutral technological progress). The index $"i"$ represents the i th sector. For the sake of later discussion, instead of calling it the *i th* "sector," we will call it the i th "firm." It is also worth noting that while the number of firms is finite, it can take a number as large as one million. Then we will rewrite the structural model into a reduced model expressed by today's and tomorrow's capital goods.

3.1 Structural model

We assume that there is one firm that produces pure-consumption goods and n firms that produce consumable capital goods that are used for both consumption and capital goods. Our model is presented as follows:

$$Maximize\sum_{t=0}^{\infty}\eta^t u(c_0(t),c_1(t),c_2(t),\cdots,c_n(t))=\sum_{t=0}^{\infty}\eta^t u(c_0(t),\mathbf{c}(t))\ (0<\eta<1) \tag{1}$$

subject to $\mathbf{k}(0) = \overline{\mathbf{k}}$

$$y_i(t) - c_i(t) + k_i(t) - \delta_i k_i(t) - (1+g)k_i(t+1) = 0 \quad (i = 1, 2, \ldots., n) \tag{2}$$

$$c_0(t) = A_0(t) f^0(k_{10}(t), k_{20}(t), \ldots, k_{n0}(t), \ell_0(t)) \tag{3}$$

$$y_t(t) = A_i(t) f^i(k_{1i}(t), k_{2i}(t), \ldots, k_{ni}(t), \ell_i(t)) \qquad (i = 1,2,\ldots., n) \tag{4}$$

$$\sum_{i=0}^{n} \ell_i(t) = 1 \qquad (t = 0, 1, 2, \ldots) \tag{5}$$

$$\sum_{j=0}^{n} k_{ji}(t) = k_i(t) \quad (i = 1, \cdots, n; t = 0, 1, 2, \cdots). \tag{6}$$

The symbols have the following meanings:

g = population growth rate $(0 \le g \le 1)$,

γ = subjective discount factor,

η = $(1+g)/(1+\gamma) \quad (g < \gamma)$,

$u(\bullet)$ = representative individual utility function,

$c_i(t) \in \mathbb{R}_+ \ (i = 0, 1, \cdots, n)$ = per capita consumption goods consumed in period t,

$y_i(t) \in \mathbb{R}_+ \ (i = 1, \cdots, n)$ = capital goods per capita production vector in period t,

$\mathbf{k}(0) \in \mathbb{R}_+^n$ = initial capital stock vector,

$f^j : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ = per capita production function of the j sector,

$k_{ij}(t)$ = capital good i used in sector j in period t,

$\ell_j(t)$ = labor input used in sector j in period t,

δ_i = depreciation rate of capital good i $(0 < \delta_i < 1)$.

We make following assumtions for avoiding further complications:

Assumption 1.

2) The utility function $u(\bullet)$ is defined on $\mathbb{R}^n_+$ so that

$$u(c_0(t), \mathbf{c}(t)) = \frac{\left(\prod_{i=0}^{n} c_i(t)\right)^{\tau}}{\tau} \quad \text{where } \tau \in (-\infty, 1) \text{ and } \tau \neq 0.$$

2) $A_i(t) = (1 + \alpha_i)^t A_i(0)$ where α_i is a rate of the Hicks neutral technical progress (or the rate of TFP growth) of the i th firm and given as $|\alpha_i| < 1$.

Assumption 2.

1) All the goods are produced non-jointly with the per capita production functions $f^i(\bullet)$ $(i = 0, 1, \cdots, n)$, which are defined on $\mathbb{R}^{n+1}$, homogeneous of degree one, strictly quasi-concave and continuously differentiable for positive inputs.

2) Any goods j $(j = 0, 1, \cdots, n)$ cannot be produced unless $k_{ij} > 0$ for some $i = 1, \cdots, n$.

3) Labor must be used directly in each sector. If the labor input of some sector is zero, the sector's output is zero.

Now dividing each output variable by its own rate of technical progress $A_i(t)$ yields the followings:

$$\tilde{y}_i(t) = \frac{y_i(t)}{A_i(t)} \quad (i = 1, \cdots, n), \qquad \tilde{c}_i(t) = \frac{c_i(t)}{A_i(t)} \ (i = 0, 1, \cdots, n).$$

The objective function is also rewritten as follows:

$$\sum_{t=0}^{\infty}\left[\frac{(1+g)}{(1+\gamma)}\right]^{t}\frac{\left(\prod_{i=0}^{n}\tilde{c}_i(t)\right)^{\tau}}{\tau}=\sum_{t=0}^{\infty}\eta^{t}u\left(\tilde{c}_0(t),\tilde{\mathbf{c}}(t)\right).$$

Assumption 3. $L(0)=1$ *and* $0<\eta<1$.

Now, under Assumption 3, the original model can be rewritten as a per capita efficiency- unit optimal problem as follows:

$$Max\sum_{t=0}^{\infty}\eta^{t}u\left(\tilde{c}_0(t),\tilde{\mathbf{c}}(t)\right)\quad \text{where } \eta=\frac{(1+g)}{(1+\gamma)},$$

$$s.t.$$

$$k_i(0)=\bar{k}_i\ (i=1,\cdots,n),$$

$$\tilde{c}_0(t)=f^{0}(k_{10}(t),k_{20}(t),\cdots,k_{n0},\ell_0(t)) \tag{7}$$

$$\tilde{y}_i(t)=f^{i}(k_{1i}(t),k_{2i}(t),\cdots,k_{ni},\ell_i(t))\quad (i=1,\cdots,n) \tag{8}$$

$$\mathbf{y}(t)-\mathbf{c}(t)+\left(\mathbf{I}-\Delta\right)\mathbf{k}(t)-(1+g)\mathbf{k}(t+1)=0, \tag{9}$$

and[1]

$$\mathbf{y}(t)=\left(\tilde{y}_1(t),\tilde{y}_2(t),\cdots,\tilde{y}_n(t)\right),\ \mathbf{c}(t)=\left(\tilde{c}_1(t),\tilde{c}_2(t),\cdots,\tilde{c}_n(t)\right),$$

$$\mathbf{k}(t)=\left(k_1(t),k_2(t),\cdots,k_n(t)\right),\ \text{and}\ \Delta=\begin{pmatrix}\delta_1 & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \delta_n\end{pmatrix},$$

$$\sum_{i=0}^{n}\ell_i(t)=1\quad and\ \sum_{j=0}^{n}k_{ji}(t)=k_i(t)\ (i=1,\cdots,n). \tag{10}$$

3.2 The reduced form model

In order to apply the turnpike theory developed by McKenzie (1983) and Scheinkman (1976), among others to the above optimization model, it is first necessary to transform

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[1] Note that all vectors are defined as row vectors. Column vectors are defined as transposed vectors, denoted by a "t" attached to the right shoulder.

the structural model into a reduced-form optimal growth model presented by a vector of initial and terminal capital stock denoted by $\left(\mathbf{k}(t), \mathbf{k}(t+1)\right)$. Let us start with the following lemma.

Lemma 1. Under Assumption 2, equations (6)-(10) except the accumulation equation (9) are summarized as the social production function $\tilde{c}(t) = T\left(\mathbf{y}(t), \mathbf{k}(t)\right)$, which is continuously differentiable in the interior of $\mathbb{R}_+^{2n}$ and concave. Due to duality, the following properties hold: $i) p_i(t) = -T_{y_i}(\mathbf{y}(t), \mathbf{k}(t))$ and $ii)\ w_i(t) = T_{k_i}(\mathbf{y}(t), \mathbf{k}(t))$.

Proof. Solving the following problem yields the result;

$$Max\ f^0(k_{10}(t), k_{20}(t), \cdots, k_{n0}, \ell_0(t)),$$

$$s.t.\ \tilde{y}_i(t) = f^i(k_{1i}(t), k_{2i}(t), \cdots, k_{ni}, \ell_i(t)), \sum_{i=0}^{n} \ell_i(t) = 1 \text{ and } \sum_{j=0}^{n} k_{ji}(t) = k_i(t) \ \ (i = 1, \cdots, n; t = 0, 1, 2, \cdots).$$

For more details, see in detail Benhabib and Nishimura (1979). ■

Subtituting the above social production function and the accumulation equation (9) into the objective function yields

$$u\left(\tilde{c}_0(t), \tilde{\mathbf{c}}(t)\right) = \frac{\left(\prod_{i=0}^{n} \tilde{c}_i(t)\right)^{\tau}}{\tau} = \frac{\left(\tilde{c}_0(t) \prod_{i=1}^{n} \Phi_i(t)\right)^{\tau}}{\tau} = \frac{\left(\tilde{c}_0(t)\right)^{\tau} \prod_{i=1}^{n} \Phi_i^{\tau}(t)}{\tau}$$

$$= \frac{\left[T\left(\mathbf{y}(t), \mathbf{k}(t)\right)\right]^{\tau} \prod_{i=1}^{n} \left[\tilde{y}_i(t) + \left(1-\delta_i\right) k_i(t) - (1+g) k_i(t+1)\right]^{\tau}}{\tau},$$

where $\Phi_i(t) = \tilde{y}_i(t) + \left(1-\delta_i\right) k_i(t) - (1+g) k_i(t+1).$

Using the above formular, we can define the reduced-form objective function as follows,

Definition.

Given $\left(\mathbf{k}(t),\mathbf{k}(t+1)\right)\in \operatorname{int}\mathbf{D}$,

$$V\left(\mathbf{k}(t),\mathbf{k}(t+1)\right)=\underset{\mathbf{y}(t)\geq\mathbf{0}}{\operatorname{Max}}\, u\left(T\left(\mathbf{y}(t),\mathbf{k}(t)\right),\prod_{i=1}^{n}\left[\widetilde{y}_i(t)+\left(1-\delta_i\right)k_i(t)-(1+g)k_i(t+1)\right]\right),$$

where $\mathbf{D}=\left\{\left(\mathbf{k}(t),\mathbf{k}(t+1)\right)\in\mathbb{R}_+^n\times\mathbb{R}_+^n : \text{There exists } \mathbf{y}(t)\geq\mathbf{0}\right.$

$\left.\text{such that the arguments of } u \text{ are non-negative}\right\}$.

The first order conditions of the above problem can be derived as follows:

$$\frac{\partial u}{\partial \widetilde{y}_i(t)}=\left(\frac{\partial u}{\partial \widetilde{c}_0(t)}\right)\bullet\left(\frac{\partial T}{\partial \widetilde{y}_i(t)}\right)+\frac{\partial u}{\partial \widetilde{c}_i}=0\ (i=1,\cdots,n; t\geq 0). \tag{11}$$

From equation (11), $\widetilde{y}_i(t)$ is the function of $\left(\mathbf{k}(t),\mathbf{k}(t+1)\right)$, and it becomes as follows:

$$\widetilde{y}_i(t)=F^i\left(\mathbf{k}(t),\mathbf{k}(t+1)\right)(i=1,\cdots,n)\Rightarrow \mathbf{y}=\mathbf{F}\left(\mathbf{k}(t),\mathbf{k}(t+1)\right).$$

Remark. It is worth noting that the above function shows the well-known Rybczynski effects in international economic theory. In connection with these effects, the following matrix may be called the "Rybczynski matrix"

$$\mathbf{Y}_t=\begin{pmatrix} \dfrac{\partial F^1(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_1(t)} & \cdots & \dfrac{\partial F^n(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_1(t)} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F^1(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_n(t)} & \cdots & \dfrac{\partial F^n(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_n(t)} \end{pmatrix}$$

and

$$\mathbf{Y}_{t+1}=\begin{pmatrix} \dfrac{\partial F^1(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_1(t+1)} & \cdots & \dfrac{\partial F^n(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_1(t+1)} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F^1(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_n(t+1)} & \cdots & \dfrac{\partial F^n(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_n(t+1)} \end{pmatrix}.$$

Then, exploiting the envelope theorem, later, those matrices are evaluated at the optimal steady state and represented using the technology matrix.

Let us define prices as follows:

Definition. $p_0(t)=\dfrac{\partial u}{\partial \widetilde{c}_0(t)}=1,\ p_i(t)=-\dfrac{\partial T}{\partial \widetilde{y}_i(t)}$ and $w_i(t)=\dfrac{\partial T}{\partial k_i(t)}\ (i=1,\cdots,n).$

Note that in terms of prices, the F.O.C. indicated by equation (1) implies that $\dfrac{\partial u}{\partial \widetilde{c}_i}=p_i(t)\ (i=1,\cdots,n)$ or in vector form it can be denoted by

$$\frac{\partial \mathbf{u}}{\partial \mathbf{c}(t)}=\mathbf{p}(t),\text{where } \frac{\partial \mathbf{u}}{\partial \mathbf{c}(t)}=\left(\frac{\partial u}{\partial \widetilde{c}_1(t)},\cdots,\frac{\partial u}{\partial \widetilde{c}_n(t)}\right)\text{and } \mathbf{p}(t)=\left(p_1(t),\cdots,p_n(t)\right).$$

We can readily rewrite our problem into the following reduced-form problem (**)

$$(**)\begin{cases}\max\limits_{\{\mathbf{k}(t)\}}\sum\limits_{t=0}^{\infty}\eta^t V(\mathbf{k}(t),\mathbf{k}(t+1)) \quad (0<\eta<1)\\ \\ s.t.\\ (\mathbf{k}(t),\mathbf{k}(t+1))\in\mathbf{D}\text{ and }\mathbf{k}(0)=\overline{\mathbf{k}}\end{cases}$$

where $V:\mathbf{D}\subset\mathbb{R}_+^{2n}\to\mathbb{R}_+,\ \mathbf{k}(t)\in\mathbb{R}^n$, and $\mathbf{k}(0)$ is a given initial stock vector.

4 Solving the optimal problem

In this section, we will work out the optimal problem (**) and derive the dynamical system that displays trajectories of the optimal solution path. The second-order cross derivatives of the objective function: $\mathbf{V}(\mathbf{k}(t),\mathbf{k}(t+1))$ will take an important role in the following sections.

4.1 Euler equations

The F.O.C. of the reduced form problem is given as the following Euler equations,

$$\mathbf{V}_{\mathbf{z}}(\mathbf{k}(t-1),\mathbf{k}(t))+\eta\mathbf{V}_{\mathbf{x}}(\mathbf{k}(t),\mathbf{k}(t+1))=\mathbf{0} \text{ for all } t\geq 0 \quad , \tag{12}$$

where $\mathbf{V}_{\mathbf{x}}(\mathbf{k}(t),\mathbf{k}(t+1))=\left[\partial V(\mathbf{k}(t),\mathbf{k}(t+1))/\partial k_1(t),\cdots,\partial V(\mathbf{k}(t),\mathbf{k}(t+1))/\partial k_n(t)\right]$,

and $\mathbf{V}_{\mathbf{z}}(\mathbf{k}(t-1),\mathbf{k}(t))=\left[\partial V(\mathbf{k}(t-1),\mathbf{k}(t))/\partial k_1(t),\cdots,\partial V(\mathbf{k}(t-1),\mathbf{k}(t+1))/\partial k_n(t)\right]$.

Based on the definition of $\mathbf{V}(\mathbf{k}(t),\mathbf{k}(t+1))$, differentiating with respect to $k_i(t)\,(i=1,\cdots,n)$

and considering the F.O.C. will yield,

$$\frac{\partial V(\mathbf{k}(t),\mathbf{k}(t+1))}{\partial k_i(t)}=\left(\frac{\partial u}{\partial \tilde{c}_0}\right)\sum_{m=1}^{n}\left(\frac{\partial T}{\partial \tilde{y}_m}\bullet\frac{\partial \tilde{y}_m}{\partial k_i(t)}+\frac{\partial T}{\partial k_i(t)}\right)+\sum_{m=1}^{n}\left(\frac{\partial u}{\partial \tilde{c}_m}\right)\left(\frac{\partial T}{\partial \tilde{y}_m}\bullet\frac{\partial \tilde{y}_m}{\partial k_i(t)}+(1-\delta_i)\delta_{im}\right)$$

$$=\left(\frac{\partial u}{\partial \tilde{c}_0}\right)\bullet\left(\frac{\partial T}{\partial k_i(t)}\right)+(1-\delta_i)\left(\frac{\partial u}{\partial \tilde{c}_m}\right)=w_i(t)+(1-\delta_i)p_i(t),$$

$$\Rightarrow \mathbf{V}_{\mathbf{x}}(\mathbf{k}(t),\mathbf{k}(t+1))=\mathbf{w}(t)+(\mathbf{I}-\Delta)\mathbf{p}(t),$$

and

$$\frac{\partial V(\mathbf{k}(t-1),\mathbf{k}(t))}{\partial k_i(t)}=\left(\frac{\partial u}{\partial \tilde{c}_0}\right)\sum_{m=1}^{n}\left(\frac{\partial T}{\partial \tilde{y}_m}\bullet\frac{\partial \tilde{y}_m}{\partial k_i(t)}\right)+\sum_{m=1}^{n}\left(\frac{\partial u}{\partial \tilde{c}_m}\right)\left(\frac{\partial \tilde{y}_m}{\partial k_i(t)}-(1+g)\delta_{im}\right)$$

$$=-(1+g)\left(\frac{\partial u}{\partial \tilde{c}_i}\right)=-(1+g)p_i(t-1)$$

$$\Rightarrow \mathbf{V}_{\mathbf{z}}(\mathbf{k}(t-1),\mathbf{k}(t))=-(1+g)\mathbf{P}(t-1),$$

where δ_{im} is the Kronecker's delta; it is one when $i=m$ or zero otherwise.

The Euler equations will also be rewritten in terms of prices as follows:

$$\begin{aligned}&\mathbf{V}_{\mathbf{z}}(\mathbf{k}(t-1),\mathbf{k}(t))+\eta\mathbf{V}_{\mathbf{x}}(\mathbf{k}(t),\mathbf{k}(t+1))\\&\quad=-(1+g)\mathbf{p}(t-1)+\mathbf{w}(t)+(\mathbf{I}-\Delta)\mathbf{p}(t)=\mathbf{0} \text{ for all } t\geq 0\end{aligned} \tag{13}$$

Remark. Note that the Euler equations feature only the supply side, that is, the prices associated with the output, and not the demand side, which is determined by the utility function of a representative consumer.

The solution path is called an ***optimal path*** and is denoted by $\left\{\mathbf{k}^{\eta}(t)\right\}_{t}^{\infty}$. Using the Euler equations, the optimal steady state path (abbreviated as OSS hereafter) is also defined as follows.

Definition. An ***optimal steady state path*** (OSS), $\mathbf{k}^{\eta}$ is an optimal path that satisfies the Euler equations (13) and $\mathbf{k}^{\eta} = \mathbf{k}(t) = \mathbf{k}(t+1)$ for all $t \geq 0$.

Note that equation (13) and the accumulation equations (9) deliver the following relation at the OSS:

$$(\gamma\mathbf{I}+\Delta)\mathbf{p}^{\eta} = \mathbf{w}^{\eta} \Rightarrow \mathbf{p}^{\eta} = (\gamma\mathbf{I}+\Delta)^{-1}\mathbf{w}^{\eta} \Rightarrow p_i^{\eta} = \frac{w_i^{\eta}}{\gamma+\delta_i} \text{ for } i = 1,\cdots,n.. \tag{14}$$

It is worth noting that equation (14) means that the prices of the i-th capital goods must equationual in the value of future rentals discounted by $(\gamma+\delta_i)$.

Linearization of the Euler equations (12) around the OSS, $(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$ is given by

$$\eta\mathbf{V}_{\mathbf{xz}}\mathbf{d}_{t+1} + (\eta\mathbf{V}_{\mathbf{xx}} + \mathbf{V}_{\mathbf{zz}})\mathbf{d}_{t} + \mathbf{V}_{\mathbf{zx}}\mathbf{d}_{t-1} = \mathbf{0} \tag{15}$$

where $\mathbf{d}_t = \mathbf{k}_t - \mathbf{k}^{\eta}$, $\mathbf{V}_{\mathbf{xx}} = \left[\frac{\partial^2\mathbf{V}(\mathbf{x},\mathbf{z})}{\partial\mathbf{x}^2}\right]$, $\mathbf{V}_{\mathbf{xz}} = \left[\frac{\partial^2\mathbf{V}(\mathbf{x},\mathbf{z})}{\partial\mathbf{x}\partial\mathbf{z}}\right]$, $\mathbf{V}_{\mathbf{zz}} = \left[\frac{\partial^2\mathbf{V}(\mathbf{x},\mathbf{z})}{\partial\mathbf{z}^2}\right]$, and each matrix is evaluated at $(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$. Later, we will consider equation (15) in detail. To avoid the complexity of the second-order cross derivatives in equation (12), we modify them with a technology matrix in the next section.

4.2 Technology matrix

Evaluating the second-order cross derivatives of equation (12) obtained directly from the Euler equations can be complicated. The use of selected technology matrices in OSS, defined below, can make us avoid this problem.

Due to the homogeneity of each firm's production, the technology matrix $\mathbf{A}$ of the economy is defined as follows,

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & & & \\ \vdots & & \mathbf{a} & \\ a_{n0} & & & \end{bmatrix} = \begin{bmatrix} a_{00} & \mathbf{a}_{0\bullet} \\ \mathbf{a}_{\bullet 0} & \mathbf{a} \end{bmatrix}$$

where $a_{0i} = \ell_i / \tilde{y}_i \; (i = 0,1,\cdots,n),\; a_{ij} = k_{ij} / \tilde{y}_j \; (i = 1,\cdots,n;\; j = 0,1,\cdots,n)$ and

$$\mathbf{a} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} .$$

The notations $\mathbf{A}^{\eta}$ and $\mathbf{a}^{\eta}$ are used to indicate that $\mathbf{A}$ and $\mathbf{a}$ are evaluated at the OSS, $\left(\mathbf{k}^{\eta}, \mathbf{k}^{\eta}\right)$. Note that Assumption 2 establishes that for all $j = 0,1,\cdots,n$, $a_{ij} > 0$ for some $i = 1,\cdots,n$ and $a_{0i} > 0$ for all i.

First, to demonstrate the existence theorem, we make the following assumption expressed in terms of the technology matrix.

Assumption 4 (Viability). For a given γ such that $0 < \eta < 1$, a chosen technology matrix $\mathbf{A}^{\eta}$ has an inverse matrix $\mathbf{B}^{\eta}$ and satisfies

$$\left[\mathbf{I} - (\gamma \mathbf{I} + \Delta)\mathbf{a}^{\eta}\right]^{-1} \gg \Theta ,$$

where Θ is a $n \times n$ zero matrix.

Remark. By the well-known equivalence theorem under the Hawkins-Simon condition and Theorem 4 of McKenzie (1960), Assumption 4 corresponds to the property that the matrix has a row dominant diagonal (d.d.) that is positive; there exists a positive column vector $\mathbf{y}^t \gg \mathbf{0}$ such that $\left[\mathbf{I}-(\gamma\mathbf{I}+\Delta)\mathbf{a}^{\eta}\right]\mathbf{y}^t \gg \mathbf{0}^t$[2]. Under Assumption 4, all the outputs turn out to be strictly positive.

McKenzie (1986, Lemma 7.1) has demonstrated the existence theorem for both an optimal and an OSS in the reduced form model. From Lemma 3 to Lemma7 in Part II of the study of Takahashi (1985) have proved that under Assumptions 1, 2 and 4, McKenzie's five conditions explained in the remark below are satisfied, and consequently the following existence theorem is established.

Existence Theorem (McKenzie). There is an optimal path $\left\{\mathbf{k}^{\eta}(t)\right\}_{t=0}^{\infty}$ from any sufficient[3] initial stock $\mathbf{k}(0)$ and a unique stationary optimal path $\mathbf{k}^{\eta}$, supported by price vectors $\mathbf{p}(t)=\eta^{t}\mathbf{q}$ in the sense that

$$V(\mathbf{k}^{\eta},\mathbf{k}^{\eta})+\eta\mathbf{q}\mathbf{k}^{\eta}-\mathbf{q}\mathbf{k}^{\eta} \geq V(\mathbf{x},\mathbf{z})+\eta\mathbf{q}\mathbf{z}-\mathbf{q}\mathbf{x} \text{ for all } (\mathbf{x},\mathbf{z})\in\mathbf{D}$$

where $\mathbf{k}(t)=\mathbf{x}$ and $\mathbf{k}(t+1)=\mathbf{z}$.

Remark. McKenzie's five conditions are as follows: 1) $\mathbf{V}(\mathbf{x},\mathbf{z})$ are defined on a convex set $\mathbf{D}$. 2) There is $\mu>0$ such that $(\mathbf{x},\mathbf{z})\in\mathbf{D}$ and $|\mathbf{x}|<\xi<\infty$ implies $|\mathbf{z}|<\mu<\infty$. 3) If $(\mathbf{x},\mathbf{z})\in\mathbf{D}$, then $\left(\tilde{\mathbf{x}},\tilde{\mathbf{z}}\right)\in\mathbf{D}$ for all $\tilde{\mathbf{x}}\geq\mathbf{x}$ and $0\leq\tilde{\mathbf{z}}\leq\mathbf{z}$. Moreover $\mathbf{V}\left(\tilde{\mathbf{x}},\tilde{\mathbf{z}}\right)\geq\mathbf{V}(\mathbf{x},\mathbf{z})$. 4) There is $\varsigma>0$ such that $|\mathbf{x}|>\varsigma$ implies that for any $(\mathbf{x},\mathbf{z})\in\mathbf{D}$, $|\mathbf{z}|<\lambda|\mathbf{x}|$ where $0<\lambda<1$.

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[2] The definitions of the symbols that express the major and minor relations between the vectors and matrices are defined in Appendix A.

[3] A capital stock $\mathbf{x}$ is called ***sufficient*** if there is a finite sequence $(\mathbf{k}(0),\mathbf{k}(1),\cdots,\mathbf{k}(T))$ where $\mathbf{x}=\mathbf{k}(0)$, $(\mathbf{k}(T),\mathbf{k}(T+1))\in\mathbf{D}$ and $\mathbf{k}(T)$ is expansible ; $\mathbf{k}(T)$ is ***expansible*** if there is $\mathbf{k}(T)$ such that $\mathbf{k}(T+1)\gg\mathbf{k}(T)$ and $(\mathbf{k}(T),\mathbf{k}(T+1))\in\mathbf{D}$. Note that the sufficiency will be assured by assuming "***Inada conditions***" in production functions.

5) There is $\left(\bar{\mathbf{x}},\bar{\mathbf{z}}\right)\in\mathbf{D}$ such that $\eta\bar{\mathbf{z}}>\bar{\mathbf{x}}$, where η is a time discount rate. Assumption 4 is required to establish 5).

From the existence theorem, there exists a unique optimal steady state output vector $\left(\tilde{c}_0^{\eta},\tilde{y}_1^{\eta},\cdots,\tilde{y}_n^{\eta}\right)$. Since $\tilde{y}_i^{\eta}=y_i^{\eta}(t)/A_i(t)\,(i=0,1,\cdots,n)$, where if $i=0$, $\tilde{y}_0^{\eta}=\tilde{c}_0^{\eta}$, it follows that $y_i^{\eta}(t)=\tilde{y}_i^{\eta}A_i(t)=(1+\alpha_i)^t A_i(0)\tilde{y}_i^{\eta}$. Therefore the original optimal series of the output $y_i^{\eta}(t)$ is growing at the rate of the firm's technical progress α_i. It is worth noting that each firm grows at the rate of the firm's TFP along the optimal steady state. In other words, in the long run, firms exhibit unbalanced growth. As a result, the firm with the higher growth rate of TFP gradually dominates the other firms along the optimal steady state. This result clearly indicates a long-run structural change.

Burmeister and Graham (1975) proved the following additional important property.

Lemma 2. When $\eta\in(0,1]$, there exists a unique $\mathbf{k}^{\eta}(\gg\mathbf{0})$ with the corresponding unique positive price vector $\mathbf{p}^{\eta}$ and the positive factor price vector $\left(w_0^{\eta},\mathbf{w}^{\eta}\right)$.

Proof. It follows from applying the same logic as the one used in Theorem 1 in Burmeister and Grahm (1975). ■

From Lemma 2, a nonsingular technology matrix is chosen for OSS, and the cost minimization and full employment conditions are established as follows:

$$\begin{cases}(1,\mathbf{p}^{\eta})=\left(w_0^{\eta},\mathbf{w}^{\eta}\right)\mathbf{A}^{\eta}\\[2ex](1,\mathbf{k}^{\eta})^t=\mathbf{A}^{\eta}\left(c_0^{\eta},\tilde{\mathbf{y}}^{\eta}\right)^t\end{cases}$$

Since $\mathbf{A}^\eta$ has an inverse matrix $\mathbf{B}^\eta$, solving the equations yields the following:

$$(***)\begin{cases} \mathbf{p}^\eta = \mathbf{w}^\eta\left(\mathbf{a}^\eta - \dfrac{1}{a^\eta_{00}}\mathbf{a}^\eta_{\cdot 0}\mathbf{a}^\eta_{0\cdot}\right) + \dfrac{\mathbf{a}^\eta_{0\cdot}}{a^\eta_{00}} = \mathbf{w}^\eta\left(\mathbf{b}^\eta\right)^{-1} + \dfrac{\mathbf{a}^\eta_{0\cdot}}{a^\eta_{00}}, \\[2ex] (\mathbf{k}^\eta)^t = \left(\mathbf{a}^\eta - \dfrac{1}{a^\eta_{00}}\mathbf{a}^\eta_{\cdot 0}\mathbf{a}^\eta_{0\cdot}\right)(\mathbf{y}^\eta)^t + \dfrac{\mathbf{a}^\eta_{0\cdot}}{a^\eta_{00}} = \left(\mathbf{b}^\eta\right)^{-1}(\mathbf{y}^\eta)^t + \dfrac{\mathbf{a}^\eta_{0\cdot}}{a^\eta_{00}}, \end{cases}$$

where $\mathbf{b}^\eta$ is the submatrix of $\mathbf{B}^\eta$ defined as follows:

$$\mathbf{B}^\eta = \left(\mathbf{A}^\eta\right)^{-1} = \begin{bmatrix} b^\eta_{00} & \mathbf{b}^\eta_{0\cdot} \\ \mathbf{b}^\eta_{\cdot 0} & \mathbf{b}^\eta \end{bmatrix}.$$

Based on these equilibrium conditions and applying the envelop theorem to (***), in the neighborhood of OSS, the following relations are established:

$$\begin{cases} \mathbf{dp} = \mathbf{dw}\left(\mathbf{b}^\eta\right)^{-1} \Rightarrow \left[\mathbf{dp}/\mathbf{dw}\right] = \left\{\left(\mathbf{b}^\eta\right)^{-1}\right\}^t \\[2ex] (\mathbf{dk})^t = \left(\mathbf{b}^\eta\right)^{-1}(\mathbf{dy})^t \Rightarrow \left[\mathbf{dk}/\mathbf{dy}\right] = \left(\mathbf{b}^\eta\right)^{-1} \text{ and } \mathbf{b}^\eta(\mathbf{dk})^t = (\mathbf{dy})^t \Rightarrow \left[\mathbf{dy}/\mathbf{dk}\right] = \mathbf{b}^\eta \end{cases} \tag*{(16)[4]}$$

where

$$\mathbf{dp} = (dp_1,\cdots,dp_n), \mathbf{dw} = (dw_1,\cdots,dw_n), \mathbf{dk} = (dk_1,\cdots,dk_n), \text{ and } \mathbf{dy} = (d\tilde{y}_1,\cdots,d\tilde{y}_n)$$

indicate vectors of differentials. Moreover note that

$$\left[\mathbf{dp}/\mathbf{dw}\right] = \begin{bmatrix} \dfrac{dp_1}{dw_1} & \cdots & \dfrac{dp_1}{dw_n} \\ \vdots & \ddots & \vdots \\ \dfrac{dp_n}{dw_1} & \cdots & \dfrac{dp_n}{dw_n} \end{bmatrix}, \left[\mathbf{dk}/\mathbf{dy}\right] = \begin{bmatrix} \dfrac{dk_1}{d\tilde{y}_1} & \cdots & \dfrac{dk_1}{d\tilde{y}_n} \\ \vdots & \ddots & \vdots \\ \dfrac{dk_n}{d\tilde{y}_1} & \cdots & \dfrac{dk_n}{d\tilde{y}_n} \end{bmatrix}, \text{ and } \left[\mathbf{dy}/\mathbf{dk}\right] = \begin{bmatrix} \dfrac{d\tilde{y}_1}{dk_1} & \cdots & \dfrac{d\tilde{y}_1}{dk_n} \\ \vdots & \ddots & \vdots \\ \dfrac{d\tilde{y}_n}{dk_1} & \cdots & \dfrac{d\tilde{y}_n}{dk_n} \end{bmatrix}.$$

It is well known in the trade theory that the relations of (***) indicate the *Stolper-*

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[4] All the vectors are defined as row vectors, and the lowercase "t" attached to the matrices and vectors indicates that they are to be transposed.

Samuelson and the *Rybczynski* effects respectively.

In the neighborhood of the OSS, the following results for the second-order cross derivatives of $T\left(\mathbf{y}(t),\mathbf{k}(t)\right)$ will be obtained. When $n=1$, Bosi et. al (2005) obtain exactly the same result as that in the following lemma. In fact the equations (18) and (21) in Bosi et. al (2005) correspond to *i)* and *ii)* of Lemma 4, respectively.

Lemma 4. The following relations hold at the OSS:

$$i)\ \mathbf{T}_{\mathbf{yk}}=-\left(\frac{1}{n}\right)\left[(\mathbf{b}^{\eta})^{-1}\right]^{t}\mathbf{T}_{\mathbf{kk}}^{t}=-\left(\frac{1}{n}\right)\left\{\mathbf{T}_{\mathbf{kk}}\left[(\mathbf{b}^{\eta})^{-1}\right]\right\}^{t},$$

$$ii)\mathbf{T}_{\mathbf{ky}}=-\left(\frac{1}{n}\right)\mathbf{T}_{\mathbf{kk}}(\mathbf{b}^{\eta})^{-1},$$

$$iii)\ \mathbf{T}_{\mathbf{yy}}=\left(\frac{1}{n^{2}}\right)\left\{(\mathbf{b}^{\eta})^{-1}\right\}^{t}\mathbf{T}_{\mathbf{kk}}(\mathbf{b}^{\eta})^{-1}.$$

Proof. First, it is easy to see that the following properties holds:

$$a)\ n\mathbf{T}_{\mathbf{yk}}=-\left[\frac{\partial\mathbf{p}}{\partial\mathbf{w}}\right]\bullet\left[\frac{\partial\mathbf{w}}{\partial\mathbf{k}}\right]\text{and b) }n\mathbf{T}_{\mathbf{yy}}=-\left[\frac{\partial\mathbf{p}}{\partial\mathbf{w}}\right]\bullet\left[\frac{\partial\mathbf{w}}{\partial\mathbf{y}}\right].$$

We can easily verify $a)$ for the case of $n=2$ as follows:

$$\begin{bmatrix}\frac{\partial T_{\tilde{y}_1}}{\partial w_1} & \frac{\partial T_{\tilde{y}_1}}{\partial w_2}\\ \frac{\partial T_{\tilde{y}_2}}{\partial w_1} & \frac{\partial T_{\tilde{y}_2}}{\partial w_2}\end{bmatrix}\bullet\begin{bmatrix}\frac{\partial w_1}{\partial k_1} & \frac{\partial w_1}{\partial k_2}\\ \frac{\partial w_2}{\partial k_1} & \frac{\partial w_2}{\partial k_2}\end{bmatrix}=\begin{bmatrix}\frac{\partial T_{\tilde{y}_1}}{\partial w_1}\bullet\frac{\partial w_1}{\partial k_1}+\frac{\partial T_{\tilde{y}_1}}{\partial w_2}\bullet\frac{\partial w_2}{\partial k_1} & \frac{\partial T_{\tilde{y}_1}}{\partial w_1}\bullet\frac{\partial w_1}{\partial k_2}+\frac{\partial T_{\tilde{y}_1}}{\partial w_2}\bullet\frac{\partial w_2}{\partial k_2}\\ \frac{\partial T_{\tilde{y}_2}}{\partial w_1}\bullet\frac{\partial w_1}{\partial k_1}+\frac{\partial T_{\tilde{y}_2}}{\partial w_2}\bullet\frac{\partial w_2}{\partial k_1} & \frac{\partial T_{\tilde{y}_2}}{\partial w_1}\bullet\frac{\partial w_1}{\partial k_2}+\frac{\partial T_{\tilde{y}_2}}{\partial w_2}\bullet\frac{\partial w_2}{\partial k_2}\end{bmatrix}=2\bullet\begin{bmatrix}T_{\tilde{y}_1k_1} & T_{\tilde{y}_1k_2}\\ T_{\tilde{y}_2k_1} & T_{\tilde{y}_2k_2}\end{bmatrix}$$
$$=2\mathbf{T}_{\mathbf{yk}}.$$

The same manipulation can be applicable to the second case. In addition, using the linear structure, the property of $n=2$ can be easily extended to a general case of $n\geq 3$. From $a)$ and (16), i) in Lemma 4 is obtained as follows:

$$n\mathbf{T}_{\mathbf{yk}}=-\left[\frac{\partial\mathbf{p}}{\partial\mathbf{w}}\right]\bullet\left[\frac{\partial\mathbf{w}}{\partial\mathbf{k}}\right]=-\left[(\mathbf{b}^{\eta})^{-1}\right]^{t}\mathbf{T}_{\mathbf{kk}}\Rightarrow\mathbf{T}_{\mathbf{yk}}=-\left(\frac{1}{n}\right)\left[\left(\mathbf{b}^{\eta}\right)^{-1}\right]^{t}\mathbf{T}_{\mathbf{kk}}$$

ii) is obtained by taking the transpose of the matrix $\mathbf{T}_{\mathrm{yk}}$. In order to obtain $iii)$, using equation (16) and b),

$$n\mathbf{T}_{\mathrm{yy}} = -\left[\frac{\partial \mathbf{p}}{\partial \mathbf{w}}\right] \bullet \left[\frac{\partial \mathbf{w}}{\partial \tilde{\mathbf{y}}}\right] = -\left[\left(\mathbf{b}^{\eta}\right)^{-1}\right]^{t} \mathbf{T}_{\mathrm{ky}} \Rightarrow n\mathbf{T}_{\mathrm{yy}} = -\left[\left(\mathbf{b}^{\eta}\right)^{-1}\right]^{t} \left\{-\left(\frac{1}{n}\right)\mathbf{T}_{\mathrm{kk}}\left(\mathbf{b}^{\eta}\right)^{-1}\right\}$$
$$\Rightarrow \mathbf{T}_{\mathrm{yy}} = \left(\frac{1}{n^{2}}\right)\left[\left(\mathbf{b}^{\eta}\right)^{-1}\right]^{t} \mathbf{T}_{\mathrm{kk}}\left(\mathbf{b}^{\eta}\right)^{-1}. \blacksquare$$

Note that $\mathbf{p} = \left(-\frac{\partial T}{\partial \tilde{y}_1}, \cdots, -\frac{\partial T}{\partial \tilde{y}_n}\right) = -\mathbf{T}_{\mathrm{y}}(\mathbf{y},\mathbf{k})$ and $\mathbf{w} = \left(\frac{\partial T}{\partial k_1}, \cdots, \frac{\partial T}{\partial k_n}\right) = \mathbf{T}_{\mathrm{k}}(\mathbf{y},\mathbf{k})$.

Then, focusing on the Euler equations (12) and differentiating $\mathbf{p} = -\mathbf{T}_{\mathrm{y}}(\mathbf{y},\mathbf{k})$ and $\mathbf{w} = \mathbf{T}_{\mathrm{k}}(\mathbf{y},\mathbf{k})$ again with respect to $\mathbf{y}$ and $\mathbf{k}$ give the following results:

$$c)\left[\partial \mathbf{p} / \partial \tilde{\mathbf{y}}\right] = -\mathbf{T}_{\mathrm{yy}},\ d)\left[\partial \mathbf{p} / \partial \mathbf{k}\right] = -\mathbf{T}_{\mathrm{yy}}\left[\partial \tilde{\mathbf{y}} / \partial \mathbf{k}\right] - \mathbf{T}_{\mathrm{yk}},$$

$$e)\left[\partial \mathbf{w} / \partial \tilde{\mathbf{y}}\right] = \mathbf{T}_{\mathrm{ky}} \text{ and f)}\left[\partial \mathbf{w} / \partial \mathbf{k}\right] = \mathbf{T}_{\mathrm{ky}}\left[\partial \tilde{\mathbf{y}} / \partial \mathbf{k}\right] + \mathbf{T}_{\mathrm{kk}}.$$

Using these results, the second-order cross derivatives of $V(\mathbf{x},\mathbf{z})$ evaluated at the OSS, $\left(\mathbf{k}^{\eta},\mathbf{k}^{\eta}\right)$ are derived as the following corollary of Lemma 4.

Corollary. The second-order cross derivatives of $V(\mathbf{x},\mathbf{z})$ evaluated at the OSS are manipulated as follows:

$$\mathbf{V}_{\mathrm{xx}} = \left[(\mathbf{\Omega}^{t})^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right]\mathbf{\Omega}^{t}\mathbf{T}_{\mathrm{kk}}\mathbf{\Omega}\left[\mathbf{\Omega}^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right], \tag{17}$$

$$\mathbf{V}_{\mathrm{xz}} = -\left(\frac{1}{n}\right)\left[\left(\mathbf{\Omega}^{t}\right)^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right]\mathbf{\Omega}^{t}\mathbf{T}_{\mathrm{kk}}\mathbf{\Omega}, \tag{18}$$

$$\mathbf{V}_{\mathrm{zx}} = -\left(\frac{1+g}{n}\right)\mathbf{\Omega}^{t}\mathbf{T}_{\mathrm{kk}}\mathbf{\Omega}\left[\mathbf{\Omega}^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right], \tag{19}$$

$$\mathbf{V}_{\mathrm{zz}} = \left(\frac{1+g}{n^{2}}\right)\mathbf{\Omega}^{t}\mathbf{T}_{\mathrm{kk}}\mathbf{\Omega}, \tag{20}$$

where $\mathbf{\Omega} = \left(\mathbf{b}^{\eta}\right)^{-1}$.

Proof.

$$\bullet\ \mathbf{V}_{\mathbf{xx}} = (\mathbf{I}-\Delta)\left[\partial \mathbf{p}/\partial \mathbf{x}\right] + \left[\partial \mathbf{w}/\partial \mathbf{x}\right] = -(\mathbf{I}-\Delta)\left[-(\mathbf{I}-\Delta)\mathbf{T}_{\mathbf{yy}} + \mathbf{T}_{\mathbf{yk}}\right] + \left[-(\mathbf{I}-\Delta)\mathbf{T}_{\mathbf{ky}} + \mathbf{T}_{\mathbf{kk}}\right]$$

$$\Rightarrow \mathbf{V}_{\mathbf{xx}} = \left(\frac{\mathbf{I}-\Delta}{n}\right)^2 \mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega} + \left(\frac{\mathbf{I}-\Delta}{n}\right)\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}} + \left(\frac{\mathbf{I}-\Delta}{n}\right)\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega} + \mathbf{T}_{\mathbf{kk}},$$

$$\Rightarrow \mathbf{V}_{\mathbf{xx}} = \left[\left(\mathbf{\Omega}^t\right)^{-1} + \left(\frac{\mathbf{I}-\Delta}{n}\right)\right]\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega}\left[\left(\mathbf{\Omega}\right)^{-1} + \left(\frac{\mathbf{I}-\Delta}{n}\right)\right].$$

$$\bullet\ \mathbf{V}_{\mathbf{xz}} = (\mathbf{I}-\Delta)\left[\partial \mathbf{p}/\partial \mathbf{z}\right] + \left[\partial \mathbf{w}/\partial \mathbf{z}\right] = -(\mathbf{I}-\Delta)\mathbf{T}_{\mathbf{yy}} + \mathbf{T}_{\mathbf{ky}}$$

$$= -\left(\frac{\mathbf{I}-\Delta}{n^2}\right)\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega} - \left(\frac{1}{n}\right)\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega},$$

$$\Rightarrow \mathbf{V}_{\mathbf{xz}} = -\left(\frac{1}{n}\right)\left[\left(\frac{\mathbf{I}-\Delta}{n}\right)\mathbf{\Omega}^t + \mathbf{I}\right]\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega} = -\left(\frac{1}{n}\right)\left[\left(\frac{\mathbf{I}-\Delta}{n}\right) + \left(\mathbf{\Omega}^t\right)^{-1}\right]\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega}.$$

$$\bullet\ \mathbf{V}_{\mathbf{zx}} = -(1+g)\left[\frac{\partial \mathbf{p}}{\partial \mathbf{x}}\right] = (1+g)\left[-(\mathbf{I}-\Delta)\mathbf{T}_{\mathbf{yy}} + \mathbf{T}_{\mathbf{yk}}\right]$$

$$= (1+g)\left[-\left(\frac{\mathbf{I}-\Delta}{n^2}\right)\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega} - \left(\frac{1}{n}\right)\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\right] = -\left(\frac{1+g}{n}\right)\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega}\left[\mathbf{\Omega}^{-1} + \left(\frac{\mathbf{I}-\Delta}{n}\right)\right].$$

$$\bullet\ \mathbf{V}_{\mathbf{zz}} = -(1+g)\left[\frac{\partial \mathbf{p}}{\partial \mathbf{z}}\right] = -(1+g)\left[-\mathbf{T}_{\mathbf{yy}}\right] = \left(\frac{1+g}{n^2}\right)\mathbf{\Omega}^t\mathbf{T}_{\mathbf{kk}}\mathbf{\Omega}. \blacksquare$$

Remark. It is worth noting that when $n=1$ and $V(k_t, k_{t+1}) = T(k_{t+1} - (1-\delta)k_t, k_t)$, the matrix $\mathbf{b}^\eta$ turns out to be a scalar and equation (17) through equation (20) of the above lemma yield exactly the same formulas as those derived by Benhabib and Nishimura (1985, p.300) as demonstrated in Appendix B.

Since the introduction of Uzawa's two-sector model, capital intensity across sectors plays an important role in stability analyses. In a two-sector model, $n=1$, the capital intensity condition can be easily defined as $k_{11}/\ell_1 > k_{10}/\ell_0 \Rightarrow a_{11}/a_{01} > a_{10}/a_{00}$, which indicates that the capital goods sector is more capital intensive than the pure-consumption goods sector. The opposite intensity is also defined similarly, which also takes an important role in our model. Inada (1971) stated that the Stolper-Samuelson condition, which has established a relationship between commodity prices and factor reward rates for the two-commodity, two-factor case, is closely related to sign patterns of the inverse matrix $\mathbf{B}^{\eta} = \left(\mathbf{A}^{\eta}\right)^{-1}$ and has generalized it to the n-commodity, n-factor case. He calls the conditions the strong Stolper-Samuelson conditions, abbreviated as the "SSS-I" ("SSS-II") condition, when the inverse matrix has positive (negative) diagonal and negative (positive) off-diagonal elements. Jones et al. (1993) derived the sufficient conditions for establishing both the SSS-I and SSS-II conditions. By adopting the generalized intensity condition used by Jones et al. (1993)[5], the capital intensities are defined in terms of technology coefficients as follows.

Definition. For $0 < \eta < 1$, the chosen technology matrix $\mathbf{A}^{\eta}$ satisfies the ***Strong Factor Intensity I Condition (SFI-I)***, if for any pair of distinct productive factors, s and r, and distinct firms s and t $(t \neq r)$,

$$\left(\frac{a_{ss}^{\eta}}{a_{rs}^{\eta}} - \frac{a_{st}^{\eta}}{a_{rt}^{\eta}}\right) > \sum_{i \neq r,s,t}^{n} \left|\frac{a_{is}^{\eta}}{a_{rs}^{\eta}} - \frac{a_{it}^{\eta}}{a_{rt}^{\eta}}\right|.$$

Similarly, the technology matrix $\mathbf{A}^{\eta}$ satisfies the ***Strong Factor Intensity II Condition (SFI-II)***, if for any pair of distinct productive factors, s and r, and distinct firms s and t $(t \neq r)$,

$$\left(\frac{a_{ss}^{\eta}}{a_{rs}^{\eta}} - \frac{a_{st}^{\eta}}{a_{rt}^{\eta}}\right) < 0 \text{ and } \left|\frac{a_{ss}^{\eta}}{a_{rs}^{\eta}} - \frac{a_{st}^{\eta}}{a_{rt}^{\eta}}\right| > \sum_{i \neq r,s,t}^{n} \left|\frac{a_{is}^{\eta}}{a_{rs}^{\eta}} - \frac{a_{it}^{\eta}}{a_{rt}^{\eta}}\right|.$$

Remark. In the trade theory, the strong Stolper-Samuelson (SSS) theorem is generally means that an increase in the price of good i leads to a simultaneous decrease in the reward rates of all factors other than factor i. However, as pointed out by Inada (1971), the SSS

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[5] Jones et al. (1993) define the capital intensity conditions in terms of factor shares; Section 3.3.2 of Nakanishi (2019) provides a comprehensive survey on the variety of generalized factor intensity conditions.

also implies that a decrease in the price of good i leads to a simultaneous increase in the reward rates of all factors other than factor i. In fact, the SFI-II condition, which is not considered by Jones et al. (1993), corresponds to this case. When $n = 1$, the SFI-I condition implies that the capital goods sector is capital intensive and the SFI-II condition implies the opposite.

Using these conditions and applying the proof of Jones et al. (1993), we propose the following lemma.

Lemma 5. If the technology matrix $\mathbf{A}^{\eta}$ satisfies the SFI-I (SFI-II) condition, the SSS-I (SSS-II) condition holds. In other words, its invertse matrix $\mathbf{B}^{\eta}$ has positive (negative) diagonal elements and negative (positive) off-diagonal elements.

Proof. From the theorem of Jones et al. (1993), under the SFI-I condition, its inverse matrix has positive diagonal and negative off-diagonal elements. On the other hand, under the SFI-II condition, we can apply exactly the same logic as in Jones et al.'s proof of the main theorem by considering the case where one price falls while all other prices remain constant. Thus, we can also obtain the second result.■

Under the GCI-I condition, the following important lemma will be proved.

Lemma 6. Under the GCI-I condition for a given η $(0 < \eta \leq 1)$, $\left[(\mathbf{b}^{\eta})^{t} - (1/n)(\mathbf{I} - \Delta)\right]$ has a ***dominant diagonal*** (abbreviated ***d.d.*** henceforth) that is positive for columns[6].

Proof. The accumulation equation (9) can be rewritten at the steady state $\mathbf{k}^{\eta}$ as follow: $\mathbf{c}^{\eta} = \mathbf{y}^{\eta} - (\Delta + g\mathbf{I})\mathbf{k}^{\eta} \geq \mathbf{0}$. Using the matrix $\mathbf{B}^{\eta}$ and the full-employment condition yields

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[6] Suppose A is an $n \times n$ matrix and its diagonal elements are positive (negative). Let there exists a positive vector $\mathbf{h} \gg \mathbf{0}$ such that $h_j \left|a_{jj}\right| > \sum_{i=1, j \neq i}^{n} h_i \left|a_{ij}\right|$, for $j = 1, \cdots, n$. Then, A is said to have a dominant diagonal that is positive (negative) for columns. A row dominant diagonal is also defined similarly. See Murata (1977).

$\mathbf{y}^{\eta} = \mathbf{k}^{\eta}\left(\mathbf{b}^{\eta}\right)^{t} + \left(\mathbf{b}_{\bullet 0}^{\eta}\right)^{t}$. Substituting this equation into the accumulation equation provides the following equation:

$$\mathbf{k}^{\eta}\left(\mathbf{b}^{\eta}\right)^{t} + \left(\mathbf{b}_{\bullet 0}^{\eta}\right)^{t} - (g\mathbf{I} + \Delta)\mathbf{k}^{\eta} \geq \mathbf{0} \Rightarrow \mathbf{k}^{\eta}\left[\left(\mathbf{b}^{\eta}\right)^{t} - (g\mathbf{I} + \Delta)\right] \geq -\left(\mathbf{b}_{\bullet 0}^{\eta}\right)^{t}.$$

Due to the fact that $\mathbf{A}^{\eta}$ satisfies the GCI-I condition, it follows that $\left(\mathbf{b}_{\bullet 0}^{\eta}\right)^{t} < \mathbf{0}$. This implies that $\left[\left(\mathbf{b}^{\eta}\right)^{t} - (g\mathbf{I} + \Delta)\right]$ has a column d.d. with $\mathbf{k}^{\eta} \gg \mathbf{0}$ and is nonsingular[7]. ■

Exploiting the above results yields the important property of the difference equation (12), originally proved by Levhari and Liviatan (1972).

Lemma 7. Provided that $\mathbf{V}_{\mathbf{zx}}^{\eta}$ is nonsingular, then if λ is the characteristic root of equation (12), then $1/\eta\lambda$ is also the characteristic root of equation (12).

Proof. The nonsingularity of the matrix $\mathbf{V}_{\mathbf{zx}}^{\eta}$ is proved as follows: from the corollary,

$$\mathbf{V}_{\mathbf{xz}} = -(1/n)\left[\left(\mathbf{b}^{\eta}\right)^{t} + (1/n)(\mathbf{I} - \Delta)\right]\left[\left(\mathbf{b}^{\eta}\right)^{-1}\right]^{t}\mathbf{T}_{\mathbf{kk}}\left(\mathbf{b}^{\eta}\right)^{-1}$$
$$\Rightarrow \det.\mathbf{V}_{\mathbf{xz}} = (-1)^{n}(1/n)\bullet\det.\left[\left(\mathbf{b}^{\eta}\right)^{t} + (1/n)(\mathbf{I} - \Delta)\right]\bullet\det.\left[\left(\mathbf{b}^{\eta}\right)^{-1}\right]^{t}\bullet\det.\mathbf{T}_{\mathbf{kk}}\det.\left(\mathbf{b}^{\eta}\right)^{-1}.$$

From Lemma 6, $\left[\left(\mathbf{b}^{\eta}\right)^{t} - (g\mathbf{I} + \Delta)\right]$ has the column d.d., which is positive. Note that the following relations hold:

$$\left[\left(\mathbf{b}^{\eta}\right)^{t} + (1/n)(\mathbf{I} - \Delta)\right] > \left[\left(\mathbf{b}^{\eta}\right)^{t} - (g\mathbf{I} + \Delta)\right].$$

This implies that $\left[\left(\mathbf{b}^{\eta}\right)^{t} + (1/n)(\mathbf{I} - \Delta)\right]$ has also the column d.d., which is positive, and it follows that $\det.\left[\left(\mathbf{b}^{\eta}\right)^{t} + (1/n)(\mathbf{I} - \Delta)\right] \neq 0$. Since $\mathbf{b}^{\eta}$ is nonsingular, $\det.\left[\left(\mathbf{b}^{\eta}\right)^{-1}\right]^{t} \neq 0$. Furthermore, $\mathbf{T}_{kk}$ is negative definite due to the strictly concavity of the social production

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[7] Any matrix A having a column d.d. is nonsingular. See the corollary of Theorem 18 of Murata (1977).

function, $T(\bullet)$, and therefore, is nonsingular. All these facts indicate the non-singularity of $\det.\mathbf{V}_{\mathbf{xz}} \neq 0$. ■

For the turnpike argument, we only concerned with the neighborhood of $\eta = 1$. From the Euler equations, the Jacobian $\mathbf{J}(\mathbf{k},\eta)$ is

$$\mathbf{J}(\mathbf{k},\eta) = \eta\mathbf{V}_{\mathbf{xx}}(\mathbf{k},\mathbf{k}) + \eta\mathbf{V}_{\mathbf{xz}}(\mathbf{k},\mathbf{k}) + \mathbf{V}_{\mathbf{zx}}(\mathbf{k},\mathbf{k}) + \mathbf{V}_{\mathbf{zz}}(\mathbf{k},\mathbf{k}).$$

Substituting the results from 1) to 4) in the corollary into the above equation and evaluating it at $\mathbf{k}^*(=\mathbf{k}^\eta$ and $\eta=1)$ yields

$$\mathbf{J}(\mathbf{k},1) = \mathbf{V}_{\mathbf{xx}}(\mathbf{k}^*,\mathbf{k}^*) + \mathbf{V}_{\mathbf{xz}}(\mathbf{k}^*,\mathbf{k}^*) + \mathbf{V}_{\mathbf{zx}}(\mathbf{k}^*,\mathbf{k}^*) + \mathbf{V}_{\mathbf{zz}}(\mathbf{k}^*,\mathbf{k}^*)$$

$$\begin{aligned} &= \left[(\boldsymbol{\Omega}^t)^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right]\boldsymbol{\Omega}^t\mathbf{T}_{\mathbf{kk}}\boldsymbol{\Omega}\left[\boldsymbol{\Omega}^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right] - \left(\frac{1}{n}\right)\left[\left(\boldsymbol{\Omega}^t\right)^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right]\boldsymbol{\Omega}^t\mathbf{T}_{\mathbf{kk}}\boldsymbol{\Omega} \\ &\quad - \left(\frac{1+g}{n}\right)\boldsymbol{\Omega}^t\mathbf{T}_{\mathbf{kk}}\boldsymbol{\Omega}\left[\boldsymbol{\Omega}^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right]\left(\frac{1+g}{n^2}\right)\boldsymbol{\Omega}^t\mathbf{T}_{\mathbf{kk}}\boldsymbol{\Omega}, \end{aligned} \quad .$$

$$\begin{aligned} \Rightarrow \mathbf{J}(\mathbf{k},1) &= \left\{\left[(\boldsymbol{\Omega}^t)^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right] - \left(\frac{1+g}{n}\right)\mathbf{I}\right\}\boldsymbol{\Omega}^t\mathbf{T}_{\mathbf{kk}}\boldsymbol{\Omega}\left\{\left[\boldsymbol{\Omega}^{-1} + \left(\frac{1}{n}\right)(\mathbf{I}-\Delta)\right] - \left(\frac{1}{n}\right)\right\} \\ &= \left[(\boldsymbol{\Omega}^t)^{-1} - \left(\frac{1}{n}\right)(\Delta + g\mathbf{I})\right]\boldsymbol{\Omega}^t\mathbf{T}_{\mathbf{kk}}\boldsymbol{\Omega}\left[\boldsymbol{\Omega}^{-1} - \left(\frac{1}{n}\right)\Delta\right]. \end{aligned}$$

From Lemma 6, the matrix $\left[\left(\mathbf{b}^*\right)^t - \left(g\mathbf{I}+\Delta\right)\right]$ has the column d.d., which is positive, and is nonsingular. Since $0 < 1/n \leq 1$ implies that $\left[\left(\mathbf{b}^*\right)^t - (1/n)\left(g\mathbf{I}+\Delta\right)\right] \geq \left[\left(\mathbf{b}^*\right)^t - \left(g\mathbf{I}+\Delta\right)\right]$. The matrix $\left[\left(\mathbf{b}^*\right)^t - (1/n)\left(g\mathbf{I}+\Delta\right)\right]$ also has the column d.d. and consequationuently is nonsingular. The fact that $\mathbf{T}_{\mathbf{kk}}\left(\mathbf{y}^*,\mathbf{k}^*\right)$ is a negative definite matrix indicates that it is also nonsingular. Thus, $\det.\mathbf{J}(\mathbf{k},1) \neq 0$. Applying the implicit function theorem to the Euler equations around the optimal steady state yields the following lemma.

Lemma 8. Suppose that the GCI-I condition holds. Then there exists a positive scalar $\overline{\eta}$

such that for $\eta \in \left[\bar{\eta}, 1\right]$, the OSS: $\mathbf{k}^{\eta}$ is a unique, continuous, and differentiable vector-valued function of η, namely, $\mathbf{k}^{\eta} = \mathbf{k}(\eta)$.

We finished preparations for introducing the turnpike theory in the next section, which was developed by McKenzie and Scheinkman.

5 The turnpike theory

In this section, we introduce the Neumann-McKenzie Facet (denoted by "NMF" for short), which plays an important role in the turnpike theory.

5.1 Value loss and the Neumann-McKenzie facet

Let us define value loss first, and then the Neumann-McKenzie facet using the value loss.

Definition. *Value loss*: θ is defined as follows:

$$\theta\left[(\mathbf{x},\mathbf{z});(\mathbf{k}^{\eta},\mathbf{k}^{\eta})\right]=\left[V(\mathbf{x},\mathbf{z})+\eta\mathbf{px}-\mathbf{pz}\right]-\left[V(\mathbf{k}^{\eta},\mathbf{k}^{\eta})+\eta\mathbf{pk}^{\eta}-\mathbf{pk}^{\eta}\right] \text{ for all } (\mathbf{x},\mathbf{z})\in\mathbf{D}.$$

The NMF is defined in the reduced-form model as follows:

Definition. *The Neumann-McKenzie Facet,* NMF henceforth, denoted by $F(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$ of OSS $\mathbf{k}^{\eta}$, is defined as follows:

$$F(\mathbf{k}^{\eta},\mathbf{k}^{\eta})=\left\{(\mathbf{x},\mathbf{z})\in\mathbf{D}:\theta\left[(\mathbf{x},\mathbf{z});(\mathbf{k}^{\eta},\mathbf{k}^{\eta})\right]=0\right\}.$$

Based on the above definition, the NMF is a set of capital stock vectors $(\mathbf{x},\mathbf{z})$ that arise from the exact same net benefit as that of the OSS when it is evaluated by the prices of the OSS. Moreover, the NMF is the projection of a flat segment on the surface of the utility function $V(\mathbf{x},\mathbf{z})$ that is supported by the price vector $\left(-\mathbf{p}^{\eta},\eta\mathbf{p}^{\eta},1\right)$ onto the

$(\mathbf{x},\mathbf{z})$ space.

Remark. This flat segment on the surface of $V(\mathbf{x},\mathbf{z})$ is originally comes from the linear homoeneity of $T(\mathbf{y},\mathbf{k})$. When $n=1$ and the utility function is $u(c)=c$, substituting the accumulation equation into the utility function yields the following reduced form of the V function:

$$V(k(t),k(t+1)) = T\big((1+g)k(t+1)-(1-\delta)k(t),k(t)\big)$$

In this case, the NMF can be illustrated in Fig 3, where the NMF is indicated by the line AB.

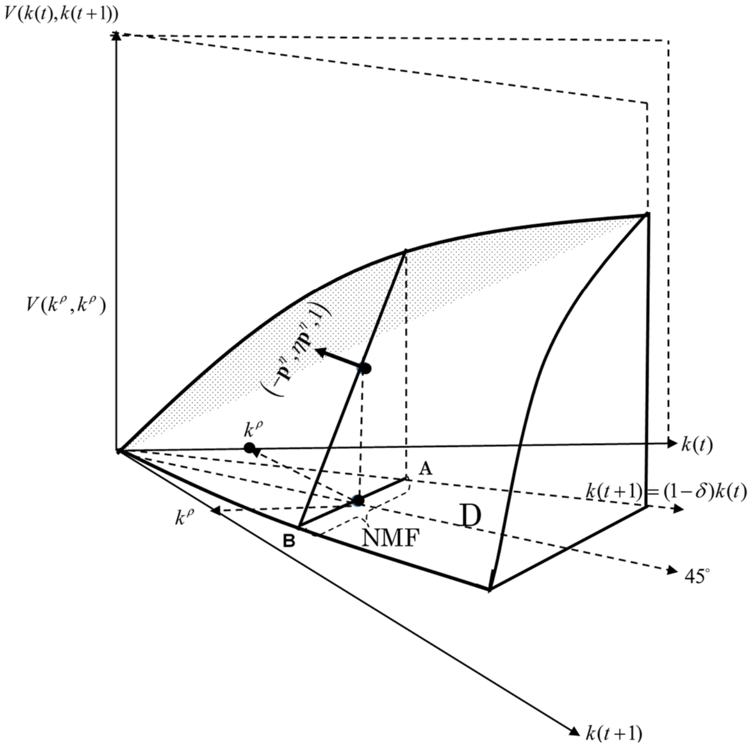

Fig.3 The Neumann McKenzie Facet

Note that due to the well-established nonsubstitution theorem, a unique technology matrix $\mathbf{A}^{\eta}$ defined previously will be chosen on the NMF. Taking advantage of this fact, Takahashi (1985) reformulated the NMF into a more tractable formula using a technology matrix as follows.

Lemma 9. $(\mathbf{x},\mathbf{z}) \in \mathrm{F}(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$ if and only if there exists $(c_0,\mathbf{y}) \geq \mathbf{0}$ such that the same technology matrix $\mathbf{A}^{\eta}$ as that of the OSS $(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$ is chosen and the following conditions hold

$$i)\ \ c_0 = c_0^{\eta},\ ii)\ \ \mathbf{y} = \mathbf{k}\left(\mathbf{b}^{\eta}\right)^{t} + \left(\mathbf{b}_{\boldsymbol{\cdot}\mathbf{0}}^{\eta}\right)^{t},\ iii)\ \mathbf{z} = \left(\frac{1}{1+g}\right)\left[\mathbf{y} + (\mathbf{I}-\Delta)\mathbf{x}\right] - \mathbf{c}^{\eta}.$$

Proof. From the definition of the NMF, it follows that

$$\left[u\left(c_0^{\eta},\mathbf{c}^{\eta}\right) + \eta\mathbf{p}\mathbf{z} - \mathbf{p}\mathbf{x}\right] = \left[u\left(c_0,\mathbf{c}\right) + \eta\mathbf{p}\mathbf{k}^{\eta} - \mathbf{p}\mathbf{k}^{\eta}\right]. \tag{19}$$

The Euler equations establish $(\gamma\mathbf{I}+\Delta)\mathbf{p}^{\eta} = \mathbf{w}^{\eta}$, and the accumulation equation yields $\mathbf{z} = \left(1/(1+g)\right)\left[\mathbf{y} + (\mathbf{I}-\Delta)\mathbf{x}\right] - \mathbf{c}^{\eta}$ and at the OSS, $\mathbf{k}^{\eta} = \left(1/(1+g)\right)\left[\mathbf{y}^{\eta} + (\mathbf{I}-\Delta)\mathbf{k}^{\eta}\right] - \mathbf{c}^{\eta}$. Substituting these relations into (15) yields,

$$\begin{aligned} &\underbrace{\left\{\left[u\left(c_0^{\eta},\mathbf{c}^{\eta}\right) - c_0^{\eta} - \mathbf{p}^{\eta}\mathbf{c}^{\eta}\right] - \left[u\left(c_0,\mathbf{c}\right) - c_0 - \mathbf{p}^{\eta}\mathbf{c}\right]\right\}}_{\text{Term }\mathbf{E}} \\ &\qquad + \overbrace{\left\{\left[c_0^{\eta} + \mathbf{p}^{\eta}\mathbf{y}^{\eta} - \mathbf{w}^{\eta}\mathbf{k}^{\eta}\right] - \left[c_0 + \mathbf{p}^{\eta}\mathbf{y} - \mathbf{w}^{\eta}\mathbf{x}\right]\right\}}^{\text{Term }\mathbf{G}} = 0 \end{aligned} \tag{20}$$

Suppose $\left(c_0^{\eta},\mathbf{c}^{\eta}\right) \neq \left(c_0,\mathbf{c}\right)$. Then, due to the strict concavity of the utility function u, the term **E** of (20) must be strictly positive. This implies that the term **G** of (20) must be negative if **E** + **G** = 0. Since $\left(\mathbf{y}^{\eta},\mathbf{k}^{\eta}\right)$ lies in the frontier of the production possibility frontier T and it is concave, the term **G** must be nonnegative. This is a contradiction. It follows that $\left(c_0^\eta, \mathbf{c}^\eta\right) = \left(c_0, \mathbf{c}\right)$ and that the term **G** must be zero. In other words, $\left(\mathbf{y}, \mathbf{k}\right)$ must also lie in the frontier of $T\left(\mathbf{y}, \mathbf{k}\right)$, and it must be supported by the same prices that support $\left(\mathbf{y}^\eta, \mathbf{k}^\eta\right)$. This also implies that in each firm, since its production function is strictly quasi-concave except for the ray from the origin, the technology used must be the same as in the OSS. In other words, the technology matrix $\mathbf{A}^\eta$ will be chosen. ■

Takahashi (1985) demonstrates the following property of the NMF.

Lemma 10. $\dim. F(\mathbf{k}^\eta, \mathbf{k}^\eta) = n-1$.

Proof. We will give an intuitive proof here. The detailed proof is provided in Lemma 17 of Takahashi (1985). From equation (20) and $i)\, c_0 = c_0^\eta$, it is clear that the degree of freedom of the NMF loses one degree. Furthermore, since the labor constraint $\sum_{i=1}^{n} \ell_i = 1$ must hold, the degree of the freedom will lose one more degree. Thus, we loose two degrees in total. Consequently, it follows that the total dimension of the NMF turns out to be $n+1-2=n-1$. ■

Remark. To explain the turnpike property, we only need to consider a property in the neighborhood of the OSS of $\eta = 1$, indicated by $\left(\mathbf{k}^*, \mathbf{k}^*\right)$, and we can make the neighborhood of the OSS as small as possible by taking η close enough to one. In other words, we can take η close enough to one so that the term E of equation (20) turns out to be zero in the neighborhood. In such a neighborhood, we can consider the allocation of labor among $(n+1)$ firms so that $\sum_{i=0}^{n} \ell_i = 1$ and the term G in equation (20) becomes zero. Therefore, in the neighborhood of the NMF, its dimension of the NMF can be regarded as n.

5.2 Neighborhood turnpike and saddle-path stability

Exploiting the value loss approach, Theorem 3 of McKenzie (1983) proves the following theorem, called the ***Neighborhood Turnpike Theorem***, without differentiability. In our differentiable case, the theorem can be rewritten as follows.

Theorem 1. Consider the following two additional conditions:

(i) **Uniform value loss:** For an $\xi > 0$ *and* $\varepsilon > 0$, there is $\bar{\theta} > 0$, such that

$|\mathbf{x}| < \xi$ *and* $d((\mathbf{x},\mathbf{z}),F(\mathbf{k}^{\eta},\mathbf{k}^{\eta})) > \varepsilon$ implies $\theta(\mathbf{k}^{\rho},\mathbf{k}^{\rho};\mathbf{x},\mathbf{z}) > \bar{\theta}$ for any η with $\bar{\eta} \le \eta \le 1$

and for any choice of $\mathbf{k}^{\rho}$, where $d((\mathbf{x},\mathbf{z}),F(\mathbf{k}^{\eta},\mathbf{k}^{\eta})) = \min_{(\mathbf{v},\mathbf{w}) \in F(\mathbf{k}^{\eta},\mathbf{k}^{\eta})} |(\mathbf{x},\mathbf{z})-(\mathbf{v},\mathbf{w})|$,

(ii) **Stability of the NMF:** The NMF of $\eta = 1$ denoted by $F(\mathbf{k}^{*},\mathbf{k}^{*})$ is stable in the sense that there are no cyclic paths on it.

Then, for any $\varepsilon > 0$ there is a $\eta' > 0$ such that $\eta' \le \eta < 1$ and $\mathbf{X}$ is sufficient implies that any optimal path $\{\mathbf{k}^{\rho}(t)\}_{t=0}^{\infty}$ for $\mathbf{k}^{\rho}(0) = \mathbf{x}$ eventually lies in the $\varepsilon(\eta) - neighborhood$ of $\mathbf{k}^{\rho}$. Furthermore, as $\eta \to 1, \varepsilon(\eta) \to 0$.

Lemma 19 in Takahashi (1985) demonstrated that the uniform value loss condition (*i*) holds by showing that the NMF, $F(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$, is a ***lower-semicontinuous correspondence***[8]. Since the proof is exceedingly long and complicated, we skip it here. We will only discuss the second condition, which is also important for a local stability argument later.

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[8] The correspondence φ is ***lower semi-continuous*** at the point $\mathbf{x}^{0}$; *if* $\mathbf{x} \to \mathbf{x}^{0}$ *as* $\nu \to \infty$ and $\mathbf{y}^{0} \in \varphi(\mathbf{x}^{0})$, then there is $\{\mathbf{y}^{\nu}\}^{\infty}$ such that $\mathbf{y}^{\nu} \to \mathbf{y}^{0}$ where $\mathbf{y}^{0} \in \varphi(\mathbf{x}^{\nu})$.

Condition (*ii*) is proved by applying the following lemma.

Lemma 11. Let us consider the following difference equation system with the equilibrium $\mathbf{x}_e = \mathbf{0}$,

$$\mathbf{x}(t+1) = (\mathbf{C} + \mathbf{I})\mathbf{x}(t),$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{C}$ is an $n \times n$ matrix. If $\mathbf{C}$ has a negative d.d. for rows, $\mathbf{C} + \mathbf{I}$ is a contraction for $\mathbf{x}(t) \neq \mathbf{0}$ with the maximum norm $\|\bullet\|$. That is, the equation system is globally asymptotically stable and the Lyapunov function is $\mathbf{V}(\mathbf{x}) = \|\mathbf{x}\|$, where $\|\bullet\|$ is defined as $\|\mathbf{x}\| = \max c_i |x_i|$ and c_i is a given set of positive numbers. Furthermore, if $\mathbf{C}$ has a positive d.d. for rows, $\mathbf{C} + \mathbf{I}$ exhibits total explosiveness for $\mathbf{x}(t) \neq \mathbf{0}$.

Proof. The first part comes from the result of Neumann (1961, pp.27-29). On the contrary, if $\mathbf{C}$ has a positive d.d. for rows, $\mathbf{C} + \mathbf{I}$ has eigenvalues with their absolute value greater than one. This comes from the fact that if $\mathbf{C}$ has a positive d.d. for rows, then its eigenvalues have a positive real part. Thus, the system is explosive; any path will diverge from equilibrium.■

Note that when $\eta = 1$, the dynamics of the NMF, $\mathbf{F}(\mathbf{k}^*, \mathbf{k}^*)$ are derived from iii) of Lemma 9 as follows:

$$\pi(t+1) = 1/(1+g)\left[\mathbf{b}^* + (\mathbf{I} - \Delta)\right]\pi(t),$$

where $\pi(t) = \left(\mathbf{x} - \mathbf{k}^*\right)^t$ and $\pi(t+1) = \left(\mathbf{z} - \mathbf{k}^*\right)^t$.

The coefficient matrix $1/(1+g)\left[\mathbf{b}^* + (\mathbf{I} - \Delta)\right]$ can be rewritten as

$$1/(1+g)\left[\mathbf{b}^* + (\mathbf{I} - \Delta)\right] = 1/(1+g)\left[\mathbf{b}^* - (g\mathbf{I} + \Delta)\right] + \mathbf{I}.$$

From Lemma 11, the following important property of the NMF will be proved in the following lemma.

Lemma 12. Provided that $\mathbf{A}^*$ satisfies the GCI-I, the NMF, $\mathbf{F}(\mathbf{k}^*,\mathbf{k}^*)$ is stable. In fact, any paths on the NMF are explosive.

Proof. From Lemma 6, the matrix $\left[(\mathbf{b}^*)^t - (g\mathbf{I}+\Delta)\right]$ has a positive d.d. for columns. Then, since $\mathbf{k}\left[(\mathbf{b}^*)^t - (g\mathbf{I}+\Delta)\right] \gg \mathbf{0} \Rightarrow \left[\mathbf{b}^* - (g\mathbf{I}+\Delta)\right]\mathbf{k}^t \gg \mathbf{0}^t$, the matrix $\left[\mathbf{b}^* - (g\mathbf{I}+\Delta)\right]$ also has a positive d.d. for rows. Thus, applying the latter part of Lemma 11, the result follows.■

Lemma 12 establishes the neighborhood turnpike. This means that any optimal path will be trapped in a neighborhood of the corresponding OSS, and the neighborhood can be taken as small as possible by making η sufficiently close to 1. This is illustrated in Fig.4.

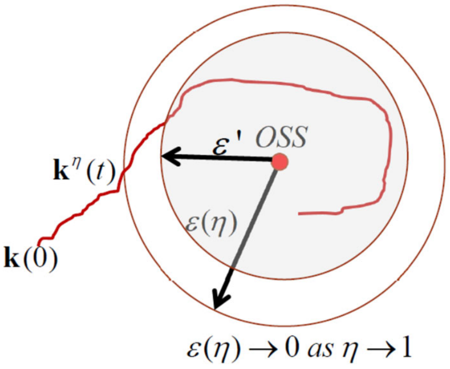

Fig.4. The neighborhood turnpike

To establish the complete turnpike property, we need to further prove the saddle-point stability in the neighborhood of the OSS. The following lemma will prove it.

Theorem 2 (Saddle-point stability). Under the GCI-I condition, for $\eta \in \left[\bar{\eta},1\right]$, the OSS, $(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$ has saddle-point stability.

Proof. From Lemma 7, the characteristic equation of equation (15) has λ as a root and it also has $1/\eta\lambda$. Furthermore, the GCI-I condition implies that the NMF, $F(\mathbf{k}^{\eta},\mathbf{k}^{\eta})$ is explosive as shown in Lemma 11. In other words, there exist n positive characteristic roots such that λ>1. By taking η close enough to one, there exists $\overset{=}{\eta} > 0$ such that for $\eta \in \left[\overset{=}{\eta}, 1\right]$, it follows that $0 < 1/\eta\lambda < 1$. The result shows that there are *n* characteristic roots with an absolute value less than 1. Therefore, the OSS satisfies the saddle-point stability, which implies that the optimal path $\left\{\mathbf{k}^{\eta}(t)\right\}_{t=0}^{\infty}$ with the initial capital stock $\mathbf{k}(0)$ in the $\varepsilon'(\eta)$-neighborhood for $\eta \in \left[\overset{=}{\eta}, 1\right]$ should converge to the corresponding OSS, $\mathbf{k}^{\eta}$. ■

This result can be illustrated in Fig.4 below. As illustrated in Fig.5, the VMF turns out to be an n-dimensional unstable linear manifold; any path on the VMF diverges from the OSS.

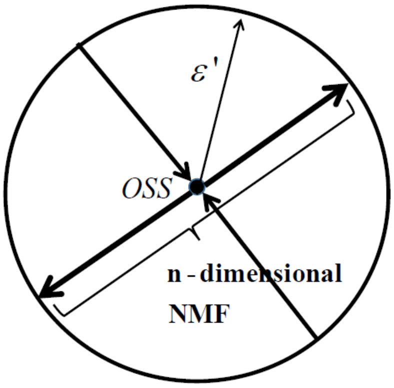

Fig.5 The saddle-path stability

Combining Theorems 1 and 2 yields the following global stability.

Theorem 3 (Global stability). There is a $\eta' > 0$ such that for $\eta \in [\eta', 1)$, any optimal path $\left\{\mathbf{k}^{\eta}(t)\right\}^{\infty}$ with the sufficient initial stocks asymptotically converges to the optimal stationary path $\mathbf{k}^{\eta}$.

Proof. Theorem 1 implies that by taking η close enough to one, there is $\bar{\eta} > 0$ such that for $\eta \in \left[\bar{\eta}, 1\right)$, any optimal path $\left\{\mathbf{k}^{\eta}(t)\right\}^{\infty}$ will be trapped in the $\varepsilon(\eta)$ -neighborhood of the corresponding OSS, $\mathbf{k}^{\eta}$, and the $\varepsilon(\eta)$ -neighborhood can be taken as small as possible by making η sufficiently close to one. On the otherhand, saddle-point stability implies that taking η close enoug to one, there exists $\overline{\overline{\eta}} > 0$ such that for $\eta \in \left[\overline{\overline{\eta}}, 1\right]$, the optimal path $\left\{\mathbf{k}^{\eta}(t)\right\}_{t=0}^{\infty}$ with the initial capital stock $\mathbf{k}(0)$ in the $\varepsilon^{'}(\eta)$-neighborhood should converge to the corresponding OSS $\mathbf{k}^{\eta}$. Taking $\hat{\eta} = \min\left(\overline{\eta}, \overline{\overline{\eta}}\right)$ yields the result. ■

Remark. We can intuitively understand the global stability as follows. Theorem 2 implies that any optimal opath will be trapped in an arbitrarily small neighborhood of the OSS by taking η close enough to one. By taking η close enoug to establish $\varepsilon(\eta) < \varepsilon^{'}(\eta)$ as shown in Fig.4-4, the optimal path should eventually jump on the stable manifold and converge to the OSS otherwise, the path would diverge from the OSS and violate optimality in the end. Furthermore, the global stability theorem implies that the optimal path of firms starting from an arbitrarily given initial sufficient stock must asymptotically converge to its own optimal steady state in which each firm is growing at a firm specific TFP growth rate.

Moreover, note that this global stability result is under the capital intensity condition that capital goods firms use more capital-intensive technology, which is in contrast to the well-known global stability result for the two-sector model, where Uzawa (1965) assumes that the pure consumption sector uses more capital-intensive technology. Indeed, Takahashi et al. (2003) provide empirical support for this case. This descripancy comes from the assumption here that the capital goods are also consumable. If capital goods are not consumable, $\mathbf{c}^{\eta} = \mathbf{0}$ and the accumulation equations yields

$$\mathbf{k}^{\eta}\left[\left(\mathbf{b}^{\eta}\right)^{t} - (g\mathbf{I} + \Delta)\right] = -\left(\mathbf{b}_{\cdot\mathbf{0}}^{\eta}\right)^{t}\cdot$$

Under the SFI-I condition, $\left[\left(\mathbf{b}^{\eta}\right)^{t}-(g\mathbf{I}+\Delta)\right]$ has a positive d.d. for column. On the other hand, under the SFI-II condition, Lemma 5 yields that $\mathbf{k}^{\eta}\left[\left(\mathbf{b}^{\eta}\right)^{t}-(g\mathbf{I}+\Delta)\right]=-\left(\mathbf{b}^{\eta}_{\bullet\mathbf{0}}\right)^{t}\ll\mathbf{0}$, and the matrix $\left[\left(\mathbf{b}^{\eta}\right)^{t}-(g\mathbf{I}+\Delta)\right]$ has a negative d.d. for columns. Applying the same logic as before, the matrix $\left[\mathbf{b}^{\eta}-(g\mathbf{I}+\Delta)\right]$ has a negative d.d. for rows and satisfies the first condition of Lemma 11. Therefore, any paths on the VMF converge to the OSS. Thus, the VMF is a stable VMF. This implies that under the SFI-II condition, the global stability is also established.

6 Conclusion and some implications

Based on the preceiding results, we can derive the following interesting implications:

i) From the existence theorem, we show that the unique optimal stationary output vector $\left(\tilde{c}_0^*,\tilde{y}_1^*,\cdots,\tilde{y}_n^*\right)$ exists. Since $\tilde{y}_i^*=y_i^*(t)/A_i(t)\ (i=0,1,\cdots,n)$, where if $i=0$, $\tilde{y}_0^*=\tilde{c}_0^*$, it follows that $y_i^*(t)=\tilde{y}_i^*A_i(t)=(1+\alpha_i)^t A_i(0)\tilde{y}_i^*$. Hence the original optimal series of the output $y_i^*(t)$ is growing at the rate of the firm's technical progress α_i. It should be noted that in the optimal steady state, each firm will grow at its own TFP growth rate, and the growth among firms is unbalanced. In other words, the firm with the highest TFP growth rate will gradually dominate the other firms in term of value added along the optimal steady state.

ii) Due to the global stability theorem, any firm's optimal path that starts from arbitrarily given initial stock should asymptotically converge to its own optimal steady state, which will grow at the firm's specific TFP growth rate.

To simplify our following discussion, let us assume that each firm uses the following Cobb-Douglas technology:

$$y_i(t) = (1+\alpha_i)^t A_i(0) k_{1i}^{\beta_{1i}} k_{2i}^{\beta_{2i}} \cdots k_{ni}^{\beta_{ni}} \ell_i^{\beta_{0i}} \quad (i = 0, 1, \cdots, n).$$

Then, the above results can be applicable when explaining Autor's superstar firm theory mentioned at the beginning of this paper. Here, it is necessary to explain the within-firm and between-industry effects simultaneously. First, let us classify the n firms into m industries. For simplicity of explanation, let m = 3: agriculture, manufacturing, and services. For example, the firm's index set: $\{0, 1, \cdots, n\}$ is divided into three subsets indicated

$$\{1, 2, \cdots, 100\} \Rightarrow \text{agriculture, } \{101, \cdots, n\} \Rightarrow \text{manufacture and } \{0\} \Rightarrow \text{service.}$$

Taking agriculture as an example, we assume that in the steady state, Firm 1 has the highest TFP growth rate (α_1) and the lowest labor income share (β_{01}). Then, in the long run, Firm 1's per capita value added (y_1) will dominate the value added of other firms in agriculture. This is the between-firm effect discussed by Autor et al. (2017). Naturally, the same logic can be applied to other industries to identify the superstar firms in each industry.

On the other hand, at the industry level, along the steady state, the aggregated macro labor income share is fully influenced by the movement of the labor income share in each industry. For example, suppose that the superstar firms in each industry are Firm 1, Firm 101, and Firm 0, respectively. These superstar firms will dominate each industry and reduce the labor income share of each industry in the long run. As a result, the aggregated macro labor income share will also decline. This is the within-industry effect discussed by Karabarbounis and Neiman (2011).

The above discussion can be summarized as follows. The superstar firm in each industry, which has the highest TFP rate, tends to dominate that industry and reduce the labor share. This is called the between-firm effect. When the same phenomenon occurs in each industry, it is called the within-industry effect, and as a result, the aggregate labor income share in the macroeconomy will decline.

Finally, let us consider policy implications of our results. Our theoretical analysis shows that, due to the within-industry effects, there exists a "superstar industry" that includes firms with extremely high TFP growth rates. This fact can be confirmed by the data shown in Fig. 6 and Fig.7 below.

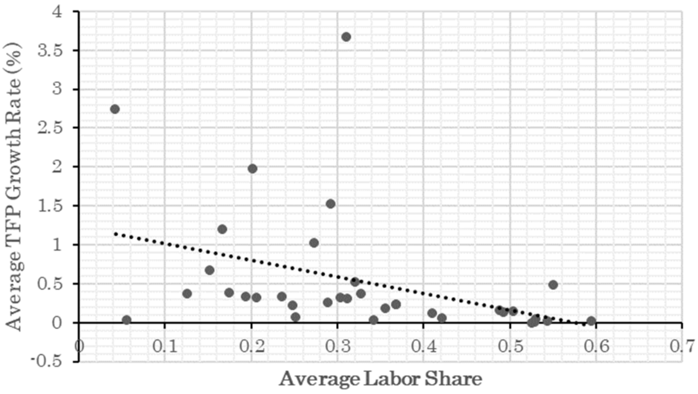

Fig.6 Average TFP Growth Rate and Labor Share from 1999 to 2010 in US:
Corr. = - 0.39

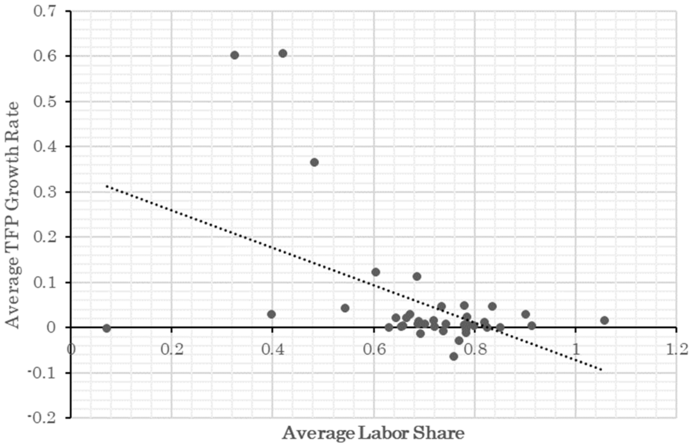

Fig.7 Average TFP Growth Rate and Labor Share from 1999 to 2010 in France:
Corr. = - 0.50

A similar relationship can be observed in other OECD countries. Although this study focuses on the superstar firm theory of the recent decline in the labor share, the actual driving forces behind the decline in the labor share are complex. The McKinzey Discussion Paper (2019, May) has re-examined the five driving forces for the US economy based on the OECD STAN database from a macro–micro perspective. Then, by ranking the five leading forces that have driven the recent capital share increase instead of those that have driven the labor share decline, the report indirectly described the main causes of the decline in the labor share as summarized in the following table.

Table 1 Contribution of respective drivers to the capital share increase

Leading Forces	**i) Supercycles and Boom-bust**	**ii) Rising and Faster Depreciation**	**iii) Superstar Effects and Consolidation**	**iv) Capital Substitution and Automation**	**v) Globalization and Labor Bargaining Power**
Weighted Contribution (%)	**33**	**26**	**18**	**12**	**11**

Since the causes of decline in the labor income share are diverse as indicated in Table 4.1, policies that address them must also be diverse.

APPENDIX A

The definitions of the symbols that express the major and minor relationships between the vectors and matrices are shown below.

(a) When $\mathbf{x}$ expresses vectors and $\mathbf{0}$ expresses zero vectors,

$\mathbf{x} \geq \mathbf{0} : x_i \geq 0$ is established for all i,

$\mathbf{x} > \mathbf{0} : x_i \geq 0$ is established for all i, and for a certain i, $x_i > 0$ is established,

$\mathbf{x} \gg \mathbf{0} : x_i > 0$ is established for all i.

(b) When $\mathbf{A}$ expresses arbitrary matrices, and $\mathbf{0}$ expresses zero matrices,

$\mathbf{A} \geq \mathbf{0} : a_{ij} \geq 0$ is established for all (i, j),

$\mathbf{A} > \mathbf{0} : a_{ij} \geq 0$ is established for all (i, j), and $a_{ij} > 0$ is established for certain (i, j),

$\mathbf{A} \gg \mathbf{0} : a_{ij} > 0$ is established for all (i, j).

APPENDX B

In this appendix, we will derive equation (17) through equation (20), when $n = 1$ and the reduced form function is given as $V(k_t, k_{t+1}) = T(k_{t+1} - (1-\delta)k_t, k_t)$ and $y_t = k_{t+1} - (1-\delta)k_t$, which is investigated in detail by Benhabib and Nishimura (1985). In this case, the Euler equation will be given as follows:

$$\frac{\partial V(k_{t-1}, k_t)}{\partial k_t} + \eta \frac{\partial V(k_t, k_{t+1})}{\partial k_t} = 0,$$

,where

$$\begin{cases} \dfrac{\partial V(k_t, k_{t+1})}{\partial k_t} = -(1-\delta)T_{y_t} + T_{k_t}, \\ \\ \dfrac{\partial V(k_{t-1}, k_t)}{\partial k_t} = T_{y_{t-1}}. \end{cases}$$

Here, T_x stands for $\partial T / \partial x$.

Due to the duality proved by Benhabib and Nishimura, and Bosi et al. (2005), the following properties hold:

$$i)\ T_{yk}=T_{ky},\ ii)T_{kk}=\frac{\partial w}{\partial k},\ iii)T_{yk}=-\frac{\partial p}{\partial k}=-\left(\frac{\partial p}{\partial w}\right)\left(\frac{\partial w}{\partial k}\right)=-b\left(\frac{\partial w}{\partial k}\right),$$
$$iv)T_{yy}=-\left(\frac{\partial p}{\partial w}\right)\left(\frac{\partial w}{\partial y}\right)=-\left(\frac{\partial p}{\partial w}\right)T_{ky}=b^2\left(\frac{\partial w}{\partial k}\right),\ v)\left(\frac{\partial p}{\partial w}\right)=b.$$

It is worth noting that b exactly corresponds to the technology matrix $\left(\mathbf{b}^{\eta}\right)^{-1}$ and our duality is generalized to the above properties.

The second derivaties can be similarly calculated and yield the following results:

$$(17)':V_{xx}=V_{k_tk_t}(k_t,k_{t+1})=(1-\delta)^2T_{y_ty_t}-(1-\delta)T_{y_tk_t}-(1-\delta)T_{k_ty_t}+T_{k_tk_t},$$
$$\Rightarrow V_{xx}=(1-\delta)^2b^2T_{k_tk_t}-(1-\delta)\left(-bT_{k_tk_t}\right)-(1-\delta)\left(-bT_{k_tk_t}\right)+T_{k_tk_t},$$
$$\Rightarrow V_{xx}=\left[b^{-1}+(1-\delta)\right]^2b^2T_{kk}.$$

$$(18)':V_{xz}=V_{k_tk_{t+1}}(k_t,k_{t+1})=-(1-\delta)T_{y_ty_t}+T_{k_ty_t},$$
$$\Rightarrow V_{xz}=-(1-\delta)b^2T_{k_tk_t}-bT_{k_tk_t}=-b^2T_{kk}[b^{-1}+(1-\delta)].$$

$$(19)':V_{zx}=V_{k_tk_{t+1}}(k_t,k_{t+1})=-(1-\delta)T_{y_ty_t}+T_{k_ty_t}=V_{xz}.$$

$$(20)':V_{zz}=V_{k_tk_t}(k_{t-1},k_t)=T_{y_ty_t}=b^2T_{kk}.$$

By setting $n=1$ in equation (17) through equation (20), we can easily obtain the similar formulas as equation (17)’ through equation (20)’. Thus we have reconfirmed our previous results.

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References

1. Autor, D., D. Dorn and L. Katz (2017) “Concentrating on the fall of the labor share,” *American Economic review: Paper & Proceedings 107*, 180-185.
2. Benhabib, J. and K. Nishimura (1979) “The Hopf bifurcation and the existence and stability of closed orbit in multisector model of optimal economic growth,” *Journal of Economic Theory 21*, 421-444.
3. Benhabib, J. and K. Nishimura (1985) “Competitive equilibrium cycles,” *Journal of*

Economic Theory 35, 284-306.

4. Bosi, S. , F. Magris and A. Venditti (2005) “Competitive equilibrium cycles with endogenous labor,” *Journal of Mathematical Economics 41*, 325-349.
5. Burmeister, E. and D. A. Graham (1975), “Price expectation and global stability in economic systems,” *Automatica 11*, 487-497.
6. Inada, K. (1971) “The production coefficient matrix and the Stolper-Samuelson condition,” *Econometrica 39*, 219-240.
7. Jones, R., S. Marjit and T. Mitra (1993) “ The Stolper-Samuelson theorem: Links to dominant diagonals,” in R. Becker, M. Boldrin, R. Jones and W. Thomson eds., *General Equationuilibrium, Growth and Trade II- the Legacy of Lionel McKenzie* (Academic Press, San Diego).
8. Karabarbounis, L. and B. Neiman (2014) “The global decline of the labor share” *Journal of Economics 129*, 61–103.
9. Levhari, D. and N. Liviatan (1972) “On stability in the saddle-point sense,” *Journal of Economic Theory 42*, 68-95.
10. McKenzie, L. (1983) “Turnpike theory, discounted utility, and the von Neumann facet,” *Journal of Economic Theory* 30, 330-352.
11. McKenzie, L. (1960) “Matrices with dominant diagonal and economic theory,” in K. Arrow, S. Karin and P. Suppes eds., *Mathematical Methods in the Social Sciences* (Stanford University Press).
12. McKenzie, L. (1986) “Optimal economic growth and turnpike theorems,” in K. Arrow and M. Intriligator, eds., *Handbook of Mathematical Economics* Vol. 3 (North-Holland, New York).
13. Murata, Y. (1977) *Mathematics for Stability and Optimization of Economic Systems* (Academic Press, New York)
14. Nakanishi, N. (2019) *The Essence of International Trade Theory* (Singapore, World Science Publishing, Singapore)
15. OECD (2003) *The Sources of Economic Growth in OECD Countries*.
16. Takahashi, H. (1985) *Characterization of Optimal Programs in Infinite Economies*, Ph.D. Dissertation (The University of Rochester).
17. Takahashi, H., K. Mashiyama and T. Sakagami (2012) “Does the capital-intensity matter?: Evidence from the post-war Japanese economy and other OECD countries” *Macroeconomic Dynamics 16, Supplement S1*, 103-116.
18. Uzawa, H. (1964) “Optimal growth in a two-sector model of capital accumulation,” *Review of Economic Studies 31*, 1-24.