

**The Appearance of Indeterminacy Paths-An
Economy with the Balanced Budget Rule and
without Depreciation**

Fujio Takada

February , 2020

Discussion Paper No.2001

GRADUATE SCHOOL OF ECONOMICS

KOBE UNIVERSITY

ROKKO, KOBE, JAPAN

The Appearance of Indeterminacy Paths – An Economy with the Balanced Budget Rule and without Depreciation

Fujio Takata

Graduate School of Economics, Kobe University;
1-1, Rokkodai-cho, Nada-ku, Kobe, 657-8501, Japan,
jikairf@clock.ocn.ne.jp.

February 6, 2020

Abstract

This paper deals with multiple movement patterns in an economy with infinite capital duration. It demonstrates what conditions are required for indeterminacy. Moreover, it shows that the capital duration is unrelated with the movement patterns.

JEL classification: C61; C62; E20; E32

Keywords: Two movement patterns; Balanced budget rule;
Labor income taxation ; Divisible labor

1 Introduction

The theme that sun spots might affect real economies is interesting. Jevons previously discussed the influence of sun spots on agricultural production and business cycles. His assertions were naturally dismissed by orthodox economists in his day, but his ideas are not unrooted fairy stories today, because we are facing the phenomenon of global warming, which will influence our economies and business cycles.

In order to analyze sun spot problems, first of all, we need to elucidate how indeterminacy is structured. Some factors have been found, such as external factors or monopolistic competition¹, increasing returns to scale, externality, and taxation.² Moreover, Kamihigashi (2002) deals with this issue in terms of externality and nonlinear discounting with respect to utility. These studies seek the causes of indeterminacy in economic conditions, that is, naturally produced economic outcomes. However, we focus on indeterminacy based on fiscal policies, such as labor income taxation and the balanced budget rule. Since we can easily manage these fiscal policies intentionally, we can induce concrete and desirable policies if necessary.

We adopt Schmitt-Urbe (1997) as our base-line model, and deduce a clear necessary condition for indeterminacy. Although our model features an economy without depreciation, we obtain the same conclusions as those in an economy with depreciation. In short, it becomes obvious that depreciation is not related to movement patterns in an economy, in our context. But Wen (2001) discusses the problem in terms of externality and depreciation.

In the next section, we deal with our base-line model. In section 3, we deduce the necessary conditions for the indeterminacy in our economy. Moreover, we verify the same conditions as those in an economy with depreciation. The fourth section is devoted to a conclusion.

2 The base line model

We describe a baseline model based on Schmitt-Urbe (1997) as

$$\max_{c_t, H_t} \int_0^{\infty} e^{-\rho t} [\log c_t - AH] dt, \quad (1)$$

subject to

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - c_t, \quad t \in \mathfrak{R}_+. \quad (2)$$

¹Benhabib and Farmer (1994).

²Guo and Lansing (1998).

Here, we denote the meaning of notations; \mathfrak{R}_+ is the set of positive reals; ρ indicates a subjective discount rate of utility, H labor inputs; and A a marginal disutility of labor, respectively. Moreover, c_t indicates consumption, K_t capital stock, τ_t labor wage income tax rates, u_t real rental rates of capital, and w_t real wage rates in the t period, respectively.

However, the assumptions in (1) and (2) have strong features. First, Schmitt-Urbe (1997) adopts the same postulations concerning labor inputs as those in Hansen (1985). Representative household agents are supposed to choose whether to work or not in every time period. Thus, their disutilities from entering the labor market are naturally positive, while those from quitting the market are null. In other words, the labor arrangements involve extensive margins, which means that in the labor supply uncertainty exists. Therefore, (1) shows the present value of an expected utility. Second, because Hansen (1985) assumes two cases, entering and quitting the labor market, the disutilities corresponding the two cases have two values, and one of them is supposed to be null. This leads to the fact that the expected disutility function really consists of one argument. In short, Hansen (1985) normalizes the disutility function so that when the agents quit the labor market, their disutilities are null. Third, Hansen (1985) assumes an arbitrary constant labor input in every time period, when the representative agents decide to work. As a result, the utility function partly consists of linear labor inputs.

But judging from common sense, due to fatigue from long labor or from aging, people can not naturally offer unlimited labor, regardless of incentives such as real wages.

Accepting what we have said above, if depreciation rates are supposed to be null, the model of Schmitt-Urbe (1997) can be modified as in (1) and (2).

Furthermore, we postulate the balanced budget rule, as described below.

$$G = \tau_t w_t H_t. \quad (3)$$

Here, G indicates government expenditure. Since the aggregate demand equalizes the aggregate supply,

$$c_t + G + K_t = F(K_t, H_t) = K_t^\alpha H_t^{1-\alpha} \quad (4)$$

prevails. Additionally, F indicates a production function. $0 < \alpha < 1$ is a constant.

3 The movements of our economy

We proceed to analyze our system and to introduce two movement types.

In light of the budget constraint in (2), as the first conditions of optimization of (1), we obtain the following system:

$$A = \Lambda_t (1 - \tau_t) w_t, \quad (5)$$

$$\frac{\dot{\Lambda}_t}{\Lambda_t} = \rho - u_t, \quad (6)$$

$$\dot{K}_t = F(K_t, H_t) - \frac{1}{\Lambda_t} - G, \quad (7)$$

and

$$G = H_t \tau_t w_t. \quad (8)$$

Additionally, $\Lambda = 1/c$.

3.1 The labor wage incomes

First, we explore how G is financed as τ s are determined. As a preparation for these analyses, we define s_{ii} and s_h , and deduce some important relationships, as below.

$$s_{ii} = \frac{K}{F} = H^{\alpha-1} K^{1-\alpha} = \left(\frac{K}{H}\right)^{1-\alpha} = \frac{\alpha}{\rho} = \frac{s_k}{\rho} = \frac{1 - s_h}{\rho},$$

and

$$s_h = \frac{Hw}{F} = wH^\alpha K^{-\alpha} = w \left(\frac{K}{H}\right)^{-\alpha} = 1 - \alpha.$$

Thus, we can express G as

$$G = \left(\frac{s_h}{s_{ii}}\right) \tau K = \left(\frac{wH}{K}\right) \tau K \quad (9)$$

Based on (9), we intend to explore G .

$$\frac{\partial G}{\partial \tau} = \left(\frac{s_h}{s_{ii}}\right) K + \left(\frac{s_h}{s_{ii}}\right) \tau \times \frac{\partial K}{\partial \tau} = \frac{G}{\tau} + \frac{G}{K} \times \frac{\partial K}{\partial \tau}, \quad (10)$$

where,

$$\frac{\partial K}{\partial \tau} = \frac{FKs_h}{F(1-\tau)s_h + F\rho s_{ii}} - \frac{K}{1-\tau}$$

$$= \frac{K s_h}{(1 - \tau) s_h + \rho s_{ii}} - \frac{K}{1 - \tau}$$

prevails. Furthermore, after some calculations, the rate of change of G regarding τ can be deduced as

$$\frac{\partial G}{\partial \tau} = G \frac{1}{\tau(1 - \tau)} \left(\left(\frac{\tau(1 - \tau) s_h}{(1 - \tau) s_h + \rho s_{ii}} - \tau \right) + (1 - \tau) \right),$$

which results in

$$\frac{\partial G}{\partial \tau} = G \times \frac{(-2\tau(s_h + \rho s_{ii}) + (s_h + \rho s_{ii}) + \tau^2 s_h)}{\tau(1 - \tau)((s_h + \rho s_{ii}) - \tau s_h)}. \quad (11)$$

Simply analyzing (11), it becomes obvious that in the range $0 < \tau < \frac{1 - \sqrt{1 - s_h}}{s_h}$, $G(\tau)$ increases, while in the range $\frac{1 - \sqrt{1 - s_h}}{s_h} < \tau < 1$, it decreases.

3.2 The steady states

Our economy is composed of (5) through (8), in which two equations showing how to move and two equations showing how to keep constant are included.

Second, in this system, we need to demonstrate the existence of steady states.

From (6),

$$\left(\frac{K}{H} \right)^* = \left(\frac{\rho}{\alpha} \right)^{-\frac{1}{1 - \alpha}} \quad (12)$$

prevails. (6) is derived from the fact that in the steady states, the marginal production of capital equals the time preference of utilities. Since the capital equipment regarding labor, $(k/H)^*$, is determined and K^* is predetermined, H is determined as

$$H^* = K^* \left(\frac{\rho}{\alpha} \right)^{\frac{1}{1 - \alpha}}, \quad (13)$$

then K as

$$K^* = \frac{(1 - \tau^*) w^*}{A \left[\rho + F_H \left(\frac{K}{H}, 1 \right) \left(\frac{H}{K} \right)^* (1 - \tau^*) \right]}. \quad (14)$$

Moreover, τ is determined as

$$G = H \tau w = \tau w^* \left(\frac{K^*}{k^*} \right), \quad k^* = \left(\frac{K^*}{H} \right). \quad (15)$$

3.3 Linearization

Here, we transform the above variables as follows:

$$\begin{aligned}\lambda_t &= \log \Lambda_t - \log \Lambda^* = \log \left(\frac{\Lambda_t}{\Lambda^*} \right), \\ k_t &= \log K_t - \log K^* = \log \left(\frac{K_t}{K^*} \right), \\ h_t &= \log H_t - \log H^* = \log \left(\frac{H_t}{H^*} \right), \\ \tilde{\tau}_t &= \log \tau_t - \log \tau^* = \log \left(\frac{\tau_t}{\tau^*} \right).\end{aligned}$$

Then, our system, (5) through (8), can be converted as

$$A = \Lambda_t (1 - \tau_t) w_t = (1 - \alpha) \Lambda_t (1 - \tau_t) H_t^{-\alpha} K_t^\alpha, \quad (16)$$

$$\frac{\dot{\Lambda}_t}{\Lambda_t} = \rho - u_t = \rho - \alpha H_t^{1-\alpha} K_t^{\alpha-1}, \quad (17)$$

$$\dot{K}_t = F(K_t, H_t) - \frac{1}{\Lambda_t} - G = -G + H_t^{1-\alpha} K_t^\alpha - \frac{1}{\Lambda_t}, \quad (18)$$

$$G = H_t \tau_t w_t = (1 - \alpha) \tau_t H_t^{1-\alpha} K_t^\alpha. \quad (19)$$

We need to linearize the above four formulae around the steady states deduced.

First, we obtain the following relationship based on (16).

$$-\frac{\alpha(H_t - H^*)}{H^*} + \frac{\alpha(K_t - K^*)}{K^*} + \frac{\Lambda_t - \Lambda^*}{\Lambda^*} - \frac{\tau^*(\tau_t - \tau^*)}{(1 - \tau^*)\tau^*} = 0,$$

which can be converted as

$$-\frac{\tau^*}{1 - \tau^*} \times \tilde{\tau}_t + s_k (k_t - h_t) + \lambda_t = 0. \quad (20)$$

Moreover, based on (17),

$$\dot{\lambda}_t = \rho s_h (k_t - h_t) \quad (21)$$

prevails, and additionally, (19) is converted as

$$\tilde{\tau}_t + s_k (k_t - h_t) + h_t = 0. \quad (22)$$

Finally, we proceed to explore how to establish the dynamic structure of (19). First, transforming (18), we obtain

$$\dot{k}_t = -\frac{G}{K_t} + H_t^{1-\alpha} K_t^{\alpha-1} - \frac{1}{K_t \Lambda_t}.$$

By a Taylor-series approximation around the steady states, we obtain

$$\begin{aligned} \dot{k}_t = & \left[\frac{G}{K^*} + (\alpha - 1) (H^*)^{1-\alpha} (K^*)^{\alpha-1} + \frac{1}{\Lambda^* K^*} \right] k_t \\ & + (1 - \alpha) h_t (H^*)^{1-\alpha} (K^*)^{\alpha-1} + \frac{\lambda_t}{\Lambda^* K^*}, \end{aligned}$$

which is crucial and eventually leads to

$$\dot{k}_t = \frac{\rho s_c}{s_k} \lambda_t + \frac{s_h}{s_{ii}} h_t + \rho k_t. \quad (23)$$

At this stage, we can say that our system consists of two equations, in which one, (20), has four arguments, λ_t , k_t , h_t and τ_t , while the other, (22), has three arguments, k_t , h_t and τ_t , and two differential equations, in which one, (23), consists of three arguments, λ_t , k_t , and h_t , while the other, (21), has three arguments, λ_t , k_t and h_t .

Here, we eliminate τ_t and h_t from the two equations. Then, by substituting the two variables into the two simultaneous differential equation systems, we obtain two simultaneous equation systems with two arguments, k_t and λ_t .

We demonstrate the procedures below. Based on (20) and (22), we obtain

$$h_t = \frac{(1 - \tau^*) \lambda_t}{s_k - \tau^*} + \frac{s_k k_t}{s_k - \tau^*}. \quad (24)$$

We should notice that the above relationship is derived from two crucial properties in the assumptions in Schmitt-Urbe (1997): First, the marginal disutility of households is constant; second, the governments follow the balanced budget rule. Consequently, the following system prevails:

$$\begin{cases} \dot{\lambda}_t = -\frac{\rho \lambda_t (1 - \tau^*) s_h}{s_k - \tau^*} - \frac{\rho k_t \tau^* s_h}{s_k - \tau^*}, \\ \dot{k}_t = \frac{\rho \lambda_t \left(s_c + \frac{(1 - \tau^*) s_h}{s_k - \tau^*} \right)}{s_k} + \frac{\rho (1 - \tau^*) k_t}{s_k - \tau^*}, \end{cases} \quad (25)$$

which obviously shows movements in our economy. So, let us explore (25) intensively. For convenience, we consider the right side in (25) as a matrix with 2 columns and 2 rows, and we denote J.

After some analyses, the trace of J is described as a function with an argument, τ^* .

$$\text{trace}(J) = \frac{\rho(\tau^* - 1)(s_h - 1)}{s_k - \tau^*}.$$

We denote this as $\Phi(\tau)$, which has properties such as $d\Phi/d\tau = \frac{\rho(s_h - 1)(s_k - 1)}{(\tau - s_k)^2} > 0$, $\Phi(0) = \rho$ and $\Phi(1) = 0$. Additionally,

$$\begin{cases} \lim_{\tau \rightarrow s_k} \Phi(\tau) \rightarrow -\infty & \text{from above,} \\ \lim_{\tau \rightarrow s_k} \Phi(\tau) \rightarrow \infty & \text{from below} \end{cases}$$

prevail.

Next, the determinant of J, $\text{Det}(J)$, is expressed as

$$\text{Det}(J) = \frac{\rho^2 s_h^2 (\tau - (1 - s_h)) \left(\tau - \frac{1 - \sqrt{1 - s_h}}{s_h} \right) \left(\tau - \frac{\sqrt{1 - s_h} + 1}{s_h} \right)}{s_k (s_k - \tau)^2}.$$

Similarly, we consider $\text{Det}(J)$ as a function with an argument, τ^* .

In our context, $\text{Det}(J) = 0$ can have three solutions regarding τ , but the solution sequence must be as follows:³

$$0 < 1 - s_h < \frac{1 - \sqrt{1 - s_h}}{s_h} < 1. \quad (26)$$

In order to elucidate (26), we replace $\text{Det}(J)$ with

$$\Pi(\tau) = -\frac{\rho^2 s_h^2 \left(\tau - \frac{1 - \sqrt{1 - s_h}}{s_h} \right) \left(\tau - \frac{\sqrt{1 - s_h} + 1}{s_h} \right)}{s_k (s_k - \tau)},$$

which has properties such as

$$\begin{cases} \Pi(0) = -\frac{\rho^2 s_h}{s_k^2} < 0, \\ \Pi(1) = -\rho^2 < 0. \end{cases}$$

Additionally,

$$\begin{cases} \lim_{\tau \rightarrow s_k} \Pi(\tau) \rightarrow \infty & \text{from above,} \\ \lim_{\tau \rightarrow s_k} \Pi(\tau) \rightarrow -\infty & \text{from below} \end{cases}$$

prevail.

³Notice that at $\tau = \frac{1 - \sqrt{1 - s_h}}{s_h}$, the labor income revenue is maximized. We express $\tau^* = \tau$ below.

Clearly, (26) implies that when $0 < \tau < 1 - s_h$, $Det(J) < 0$, while $1 - s_h < \tau < \frac{1 - \sqrt{1 - s_h}}{s_h}$, $Det(J) > 0$, and that when $\frac{1 - \sqrt{1 - s_h}}{s_h} < \tau < 1$, $Det(J)$ becomes negative.

In order to establish this, s_k or s_h must satisfy a condition, which can be obtained by solving the previous inequity, (26). The solution is a necessary condition for indeterminacy. The condition in terms of α is approximately expressed as

$$0 < \alpha < 0.56984.$$

The above condition requires a low share of capital in output. The reason (26) is a necessary condition becomes obvious by examining the relationships between $trace(J)$ and $Det(J)$.

Figure 1 shows the necessary and sufficient conditions for indeterminacy, the set $\{\alpha, \rho, A, G\}$ that satisfies $\alpha < \tau^* < \frac{1 - \sqrt{\alpha}}{1 - \alpha}$. In this situation, we should notice that τ^* depends on the set with all four elements, α, ρ, A , and G . Although τ^* is affected by α , the value of τ^* must be larger than α and smaller than $(1 - \sqrt{\alpha})/(1 - \alpha)$, for arbitrary ρ, A and G . This implies the existence of a necessary condition for indeterminacy in terms of α , which was deduced.

Here, we cite examples.

Case (a): $A=1; \alpha = 0.4; \rho = 0.01; G=1$.

In this case, $\tau^* = 0.273$ holds. So a saddle path occurs, because $(1 - \sqrt{\alpha})/(1 - \alpha) = 0.612$.

Case (b): $A=1; \alpha = 0.4; \rho = 0.01; G=1.5$.

In this situation, $\tau^* = 0.497$ holds, so indeterminacy occurs.

Based on the information analyzed so far, we can depict $Det(J)$ and $trace(J)$ as follows:

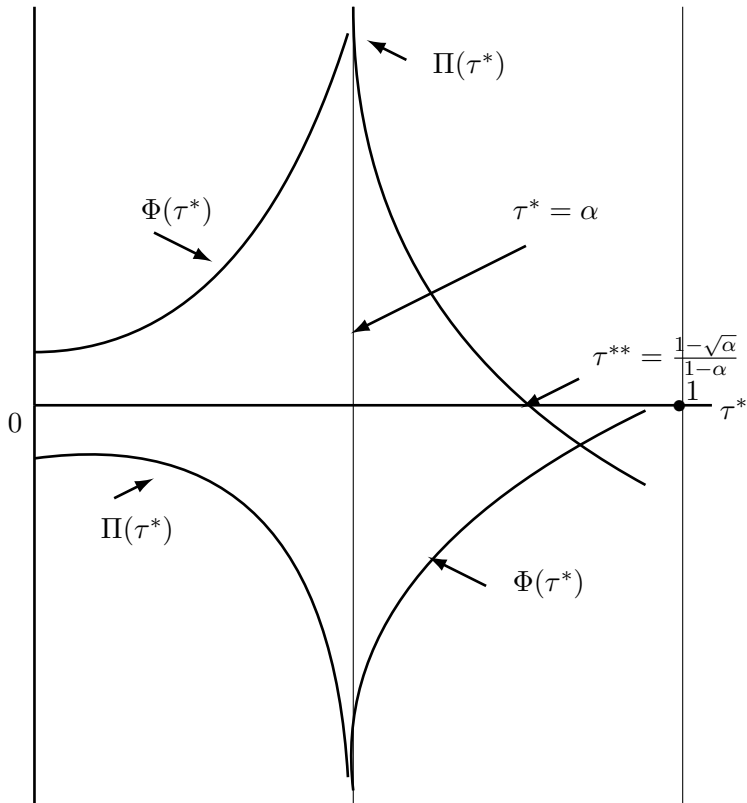


Figure 1: Given $\alpha < \tau^* < \tau^{**}$, indeterminacy paths exist; when $0 < \tau^* < \alpha$, or $\tau^{**} < \tau^* < 1$, a saddle path occurs.

4 Conclusion

We have discussed indeterminacy problems, assuming a labor arrangement with extensive margins. Households decide to work or not over whole lifetime periods. On the other hand, governments follow the balanced bud-

get rule, while firms make decision to gain maximum profits in every period. As a result, their decisions are shortsighted and passive. Their behavior contrasts directly with that of households, resulting in a focus on households, especially in terms of labor supply.

We have demonstrated that if governments adopt a balanced budget rule, indeterminate paths can occur, in an economy with capital composed of infinite duration, and that a necessary condition in terms of the ratio of capital distribution to outputs accounts for the phenomenon.

In line with this, if we seek the anticipations of economic agents as one source of sun spots as in Farmer (2002), our anticipations will be self-fulfilled.

References

- [1] Attanasio, P. O., (2015), “Frank Ramsey’s Mathematical Theory of Saving,” *The Economic Journal*, Vol. 125, No. 583, 269-277.
- [2] Benhabib, J. and Farmer, R. F., (1994), “Indeterminacy and Increasing Returns,” *Journal of Economic Theory*, 63, 19-41.
- [3] Farmer, E. A., (2002), *Macroeconomics of Self-fulfilling Prophecies, 2nd edition* (Cambridge, MA: MIT Press)
- [4] Hansen, G. D., (1985), “Indivisible Labor and the Business Cycle,” *Journal of Monetary Economics*, 16, 309-327.
- [5] Kamihigashi, T., (2002), “Externality and Nonlinear Discounting: Indeterminacy,” *Journal of Economic Dynamics and Control*, 26, 141-169.
- [6] Mino, K., (2017). *Growth and Business Cycles with Equilibrium Indeterminacy* (Tokyo: Springer) *Advances in Japanese Business and Economics*, 13.
- [7] Mulligan C., (1999), “Microfoundations and Macro Implications of Indivisible Labor,” *NBER Working Paper*, No. 7116.
- [8] Ramsey, F. P., (1928), “A Mathematical Theory of Saving,” *The Economic Journal*, 38, No. 152 (Dec.), 543-559.
- [9] Rogerson R., (1988), “Indivisible Labor, Lotteries and Equilibrium,” *Journal of Monetary Economics*, 21, 3-16.
- [10] Schmitt-Grohé, S. and Uribe, M., (1997), “Balanced-Budget Rules, Distortionary Taxes and Aggregation Instability,” *Journal of Political Economy*, Vol. 105, No. 5, 976-1000.
- [11] Takata, F., (2017), “Two Steady States and Two Movement Patterns under the Balanced Budget Rule-An Economy with Divisible Labor,” *Discussion Paper No.1702, January 2017, Kobe University*.
- [12] Wen, Y., (2001), “Understanding Self-Fulfilling Rational Expectations Equilibria in Real Business Cycle Models,” *Journal of Economic Dynamics and Control*, 25, 1221-1240.