Relume: A fractal analysis for the US stock market

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Relume: A fractal analysis for the US stock market

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Abstract

We present a fractal market hypothesis for US stock prices.

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Fractals are the concern of a new geometry, whose primary object is to describe the great variety of natural structures that are irregular, rough or fragmented, having irregularities of various sizes that bear a special 'scaling' relationship to one another. (Mandelbrot, et al., 1984)

1 Introduction

The recent financial crisis has cast doubt on traditional macroeconomics repeatedly. During the crash, international financial markets crashed simultaneously, and the fragility of numerous financial institutions and private banks was exposed. However, if agents in the real world had been rational and therefore had perfect foresight, the crisis would have been less severe.

Kindleberger (1977, Chap. 5) also criticized the rational expectations hypothesis from a historical perspective: the rational expectations response immediately to shocks on the long-run equilibrium, but in reality, expectations change more slowly.

The efficient market hypothesis also fails to capture certain aspects of financial markets' behavior. The hypothesis suggests that asset prices follow a random walk, which implies that shocks to asset prices are not persistent because the current price contains all available information and responds immediately to new shocks through the rational expectations of agents.

Fractal geometry, established by Benoit B. Mandelbrot, addresses the problem of market efficiency. Mandelbrot (1971) sheds light on the long-memory property of asset

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prices, which he labeled the "Joseph Effect"— a shock that occurred in the distant past does not disappear, but rather continues to affect current prices. The intuitive definition of *fractals* provided by Mandelbrot (1986) is: *A fractal is a shape made of parts similar to the whole in some way.* Consistent with this definition, we examine the *self-affinity* (fractal dimension) of the S&P 500 stock price index in the US stock market.[12]

A translation that scales time and distance by different factors is called affine, and curves that reproduce themselves in some sense under an affine transformation are called *self-affine* (Feder, 1988, Chap. 9). In general, stock prices are an example of this.[3]

This paper provides new perspectives on the US stock market using fractal geometry.[4] We emphasize that US stock prices are fractal. To this end, we depend on rescaled range analysis to determine the fractal dimension of the stock price index.

This paper proceeds as follows. Section 2 describes our empirical methods for identifying the fractal dimension. Section 3 presents our empirical findings. Section 4 concludes.

2 Methodology

This section provides a brief summary of *Hurst's rescaled range analysis* for specifying the fractal dimension D of the US stock price index.

We describe the stock price return for any given time t as the stochastic process $x(t) \in \big(x(1), ..., x(T)\big)$. Let $\langle\rangle$ be an ensemble average over window size τ:

$$\langle x(t) \rangle = \frac{1}{\tau} \sum_{\tau(l-1)+1}^{\tau l} x(t), \tag{1}$$

for $l = 1 : \text{ceil}(T/\tau)$. Then, let $x(t, \tau)$ be the *accumulated* dispersion for $x(t)$ from the average:

$$x(t,\tau) = \sum_{t=1}^{T} \big(x(t) - \langle x(t) \rangle\big), \tag{2}$$

and let $R(\tau)$ be the range for $x(t, \tau)$ over the window size τ:

$$R(\tau) = \max_{\tau(l-1)+1 \le t \le \tau l} x(t,\tau) - \min_{\tau(l-1)+1 \le t \le \tau l} x(t,\tau). \tag{3}$$

Clearly, the range R is an increasing function of τ. Also, let $S(\tau)$ be the standard deviation from the ensemble average by the window τ:

$$S(\tau) = \sqrt{\frac{1}{\tau} \sum_{\tau(l-1)+1}^{\tau l} \big(x(t) - \langle x(t) \rangle\big)^2}. \tag{4}$$

[1]Fractal geometry is related closely to chaos theory because the shape of a strange attractor is a type of fractal. See Branch and MacGough (2010) for beautiful bifurcations of the New Keynesian system with dynamic predictor selection of the agents.

[2]See Booth et. al., (1982) for the fractal analysis of exchange rates.

[3]I would like to thank Shinichi Sato and Jun-ichi Wakita for raising this point.

[4]Branch and Evans (2011, 2013) develop sophisticated models of stock price dynamics with adaptive learning algorithms.

Hurst (1951, 1955) uses the dimensionless ratio R/S,[5] and found that the observed rescaled range, $R(\tau)/S(\tau)$, is very well described by *the power law* as:

$$R(\tau)/S(\tau) \propto \tau^H, \qquad 0 < H < 1. \tag{5}$$

Here, H corresponds to *the Hurst exponent*. We can deduce the fractal dimension D by the relationship, $D = 2 - H$ (Mandelbrot, 1985).[6] Note that the mean and the standard deviation lose their meanings in the case of the power law distribution because the power distribution considers unexpected large fluctuations of the series relative to the bell-shaped distribution.

We distinguish three cases using H as follows.

(i) *Antipersistence* ($0 < H < 0.50$): very noisy process. A time series that was increasing (decreasing) in the past will decrease (increase) today.

(ii) *Random walk* ($H = 0.50$): ordinary Brownian motion frequently used in many applications. On average, past shocks have no effects on the series today.

(iii) *Long memory* ($0.50 < H < 1.00$): superpersistent behavior. The noise in the distant past never disappears and therefore continues to affect the dynamics forever.

The noise in the system decreases as H increases, implying that the trend in the variable is increasing. It is possible in long memory case that a previously increasing trend may suddenly become a decreasing trend in the future. Apart from the random walk hypothesis, the fractal market hypothesis, by its nature, considers the possibility of a large crash in stock markets, as is seen frequently in reality.

3 Empirical results

This section shows the estimated Hurst exponent for determining the fractal dimension of US stock prices. To this end, weekly values of the S&P 500 stock price index from the first week of September 2006 to the first week of September 2016 are used in log-differential form.[7][8] In our regression, we use nonparametric bootstrapping to calculate 95% confidence intervals for the Hurst exponent because it is impossible to define the variance of the regressand in the long-memory case.

3.1 Benchmark results

Figure 1 presents the fitted lines for equation (5) in logarithms and the scatter plot of the realized values for the rescaled range of S&P growth rates against the corresponding time windows. Note that the fitted lines generated by 5555 bootstrapping contain almost all realized values, which justifies our regression.

[5] Hurst investigates many natural phenomena such as the storage capacity of the Nile River.

[6] I would like to thank Tamàs Vicsek for making this point.

[7] We use all available data from the Federal Reserve Economic Data (FRED) database.

[8] All results in this paper are robust with respect to data frequency and index type. Results for the Dow Jones Industrial Average index are available upon request.

The estimated value of the Hurst exponent is reported in Table 1. As shown in the table, the Hurst exponent is 0.7250, and therefore the stock price return is fractal rather than a random walk. Furthermore, note that the 95% confidence intervals contain the range of $0.50 < H < 1.00$, so that our insist is prominent.[9]

3.2 Five-year cycle of US stock prices[10]

The high Hurst exponent indicates that stock prices are *strongly non-Gaussian*. However, in Figure 1, we find that the fitted line does not perfectly replicate the scatter plot: the plots deviate from the line for low values of τ. This suggests that the slope of the fitted line, H, may vary by τ. According to Feder (1988, Chap. 11), there exists a cycle in which the fitted value of H changes substantially. To show the dependence of H on τ, we implement sequential OLS by τ.

Figure 2 presents the Hurst exponent by sequential regression over all time spans.[11] The Hurst exponent is unstable and records values smaller than 0.7250 over time spans of less than five years, while it converges to the estimated value thereafter.[12] This result suggests that the stock price is less persistent and fickle during short horizons but becomes stable over horizons of longer than five years.

In sum, we found that US stock prices are fractal.

4 Conclusion

This paper revives fractal analysis in macroeconomics. Our results suggest that US stock prices are fractal, rather than random walks.

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[9] This result is amazing because Hurst's original estimates for H using his random walk simulation is 0.714 ± 0.091.

[10] I would like to thank Jens Feder because this subsection has been improved by his comments.

[11] We use the empirical distribution of the Hurst exponent to calculate the 95% confidence intervals, because the minimum sample size of the recursive regression is only three. Bootstrapping is very fragile in small samples.

[12] The unstable region of H less than five years corresponds to the recent financial crisis.

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Figures and Tables

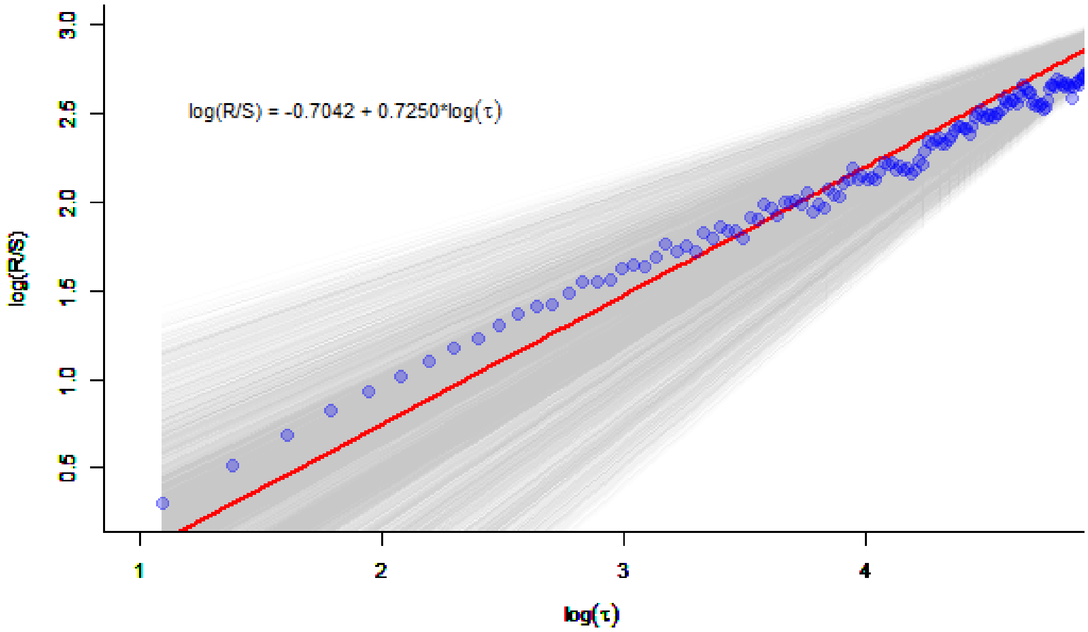

Figure 1: Fitted and realized values for the rescaled range

Note: (i) $\log(R/S)$ denotes the rescaled range for the S&P growth rates. (ii) Gray lines are generated by regressions using 5555 bootstrapping.

Table 1: Hurst exponent for the S&P 500

Hurst exponent (H)	
0.7250	
(0.4953)	(1.1022)

Note: (i) Figures in parentheses denote bootstrapped 95% confidence intervals.

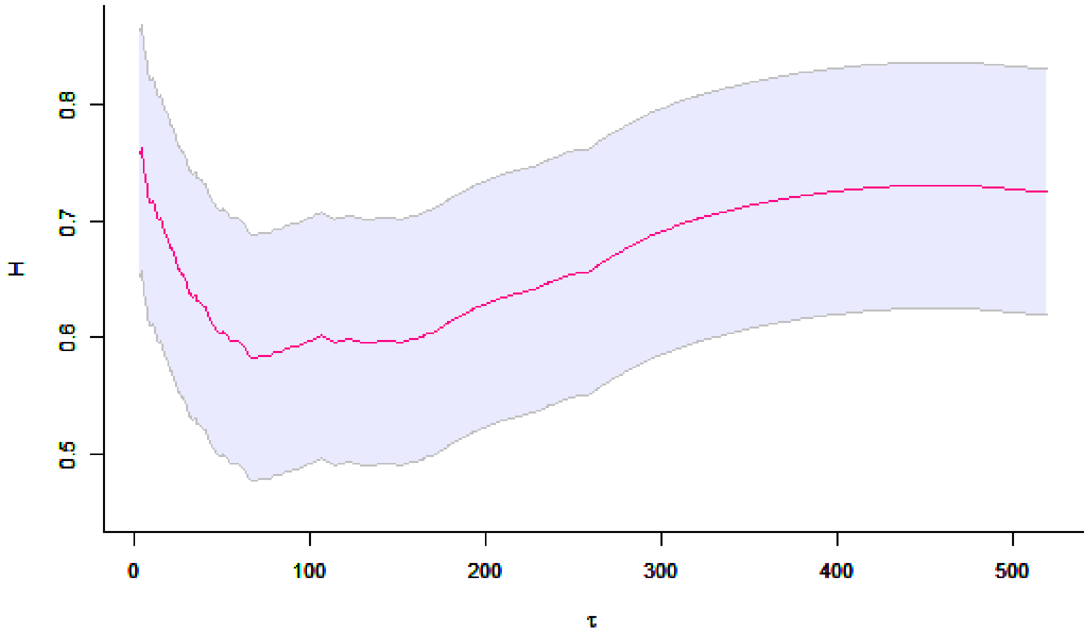

Figure 2: Sequential estimates for the Hurst exponent

Note: Shaded area denotes 95% confidence interval for H.