

Two Movement Patterns under the Balanced
Budget Rule – Further Results

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Two Movement Patterns under the Balanced Budget Rule—Further Results*

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Abstract

When governments levy taxes on labor income on the basis of a balanced budget rule, two steady states in an economy exist, which can cause two movement patterns, namely, indeterminacy paths and a saddle path. Many economists deal with this issue based on indivisible labor. On an general assumption of increasing marginal disutility of labor, that is, divisible labor, however, we demonstrate that for indeterminacy, an upper limitation concerning the share of capital in output is needed.

JEL classification: E13; E21; E32; E62;

Keywords; Two movement patterns; Balanced budget rule; Labor income taxation; Divisible labor

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1 Introduction

In a previous paper, Takata (2013), we demonstrated that there exist two steady states in an economy governed by a balanced budget rule, in which labor income taxation is assumed. And in this situation, we anticipated that corresponding to the existence of two steady states, there also exist two different movement patterns, namely, saddle paths and indeterminate ones. From the viewpoint of this perspective, in this subsequent paper, we aim to explore our anticipation.

There is a lot of literature on the issue of indeterminacy. Originally, Black (1974) and Brock (1974) point out a possibility of indeterminacy in a dynamic monetary model. Since then, many economists have located the causes of this indeterminacy in external factors in production or monopolistic competition. In line with this, we can list Benhabib and Farmer (1994), Guo and Lansing (1998), Wen (2001),¹and Kamihigashi (2002). Obviously, these authors mainly seek the causes of indeterminacy in economic conditions, that is, in naturally produced economic outcomes.

However, we would like to discuss indeterminacy from another viewpoint, that is, in relation to fiscal policy, as a product only of artificial determinants. In this context, fiscal policy specifically means labor income taxation and a balanced budget principle. In the same context, we can list Schmitt-Grohé and Uribe (1997) and Anagnostopoulous and Giannitsarou (2012). They examine this hot issue in an infinite horizon model and on the basis of a government balanced budget rule.

However, they assume indivisible labor, following Hansen (1985) and Rogerson (1988). This assumption asserts that a change in labor is caused only by entering or quitting the labor market, and that the utility function with regard to labor is linear. Kamihigashi (2000) is also included in this category.

However, in contrast, another reasonable idea is that all households choose their labor supply according to adjustments along an intensive margin, that is, with variations in utilization. In this case, all households work the same amount. This means divisible labor.

In light of the above argument, we deal with two movement patterns based on a new model, which includes Schmitt-Grohé and Uribe (1997) as

¹These papers discuss a one-sector model. On the other hand, Benhabib and Farmer (1996) and Guo and Harrison (2001) discuss indeterminacy in a two-sector model.

a special case. In this article, we deal with labor income taxation, in which the tax rates on labor income are flexible and, correspondingly, government expenditure is exogenously constant, under the balanced budget rule.

As a result, we conclude that an explicit labor supply function is deduced, and that this system causes nonlinear dynamics which can cause the coexistence of indeterminacy paths and a saddle path under the same economic fundamentals. And we demonstrate some conditions for a possible indeterminacy, which occurs at a lower range of labor tax rates in a steady state than that corresponding to maximum tax revenue. One of the conditions is the value of the tax rates where indeterminacy occurs, which consists of two key fundamentals, the share of capital in output and the elasticity of labor supply with respect to wages. Moreover, in order for indeterminacy to occur, the share of capital in output has to be less than approximately 0.5698. In other words, when indeterminacy occurs, the ratio of capital in output to that of labor in output must be less than 1.32, approximately. This is explained in *Theorem 1*. (See subchapter 2 in Chapter 6)

We organize this article as follows: In the next section, as premises, we introduce the framework and approach of the paper, in which the labor supply is deduced explicitly. And we explain a dynamic system, which was established in the previous paper, Takata (2013). In what follows, in section 3, we transform the system into a linear one. In section 4, we confirm that our system, which on appearance consists of three variables, can be controlled by just two variables. Subsequently, in section 5, determination of initial conditions is discussed, especially from the viewpoint of a state variable and adjoint ones. Then, the existence of the two different movements and some conditions for indeterminacy are deduced, in section 6. Finally, section 7 is devoted to a conclusion. After the conclusion, an appendix follows.

2 Premises

Before we discuss our main issues, we offer the following assumptions and results, obtained in Takata (2013).

An economy consists of three agents, namely households, firms and a government. Households and firms are each assumed to be represented by an agent. A representative household possesses capital and loans it to a firm in exchange for rental fees, and additionally offers its labor to a firm. The labor-population is constant and households are identical. A household faces

a dynamic problem in which it determines an infinite scenario of work and consumption under budget constraints over time.

In this situation, the three agents behave as follows:

First, a household is assumed to live forever² and to maximize lifetime utility. More concretely, it considers labor tax rates, τ_t , real wage rates, w_t , and real rental rates of capital, u_t , as given, and solves the following dynamic problem:

$$\max_{c_t, H_t} \int_0^{\infty} e^{-\rho t} \left[\log c_t - \frac{H_t^{1-\chi}}{1-\chi} \right] dt,$$

subject to

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - c_t, \quad t \in \mathfrak{R}_+.$$

Here, \mathfrak{R}_+ is the set of positive reals. Additionally, the notations regarding an instantaneous utility function show c_t as consumption, H_t as working hours, and ρ as a subjective discount rate of utility. Here, χ indicates a constant parameter and is negative,³ and K_t capital stock at t .

Second, the government observes a balanced budget rule, which can be expressed as $G = \tau_t w_t H_t$. Of course, G shows government spending.

Third, a firm aims to maximize its profit based on a production function $F(K_t, H_t) = K_t^\alpha H_t^{1-\alpha}$. Here, α is a constant satisfying $0 < \alpha < 1$.

We assume that the markets for labor, capital, and output are fully competitive. Additionally, the output price is assumed to be 1, which shows that our system is measured in terms of outputs.

In this situation, a transversality condition leads to $\lim_{t \rightarrow \infty} \frac{k_t}{c_t} e^{-\rho t} = 0$, where $k_t = K_t/H_t$.

Given these propositions, it is proved that the dynamic system of our economy consists of the following three differential equations:⁴

$$\frac{\dot{c}_t}{c_t} = u_t - \rho = \alpha k_t^{-(1-\alpha)} - \rho,$$

²We, however, assume that a household eventually dies at infinity, and that capital stock can be transformed into consumption with perfect substitution.

³Schmitt-Grohé and Uribe (1997), Kamihigashi (2000) and Angnostopoulos and Giannitsarou (2013) are all based on Hansen (1985) with regard to labor supply. This is representative as $\chi = 0$ in our model.

⁴See Takata (2013)

$$\begin{aligned}\frac{\dot{k}_t}{k_t} &= \frac{(\alpha + (1 - \alpha)(1 - \tau_t))((1 - \chi)\tau_t + \chi) - \alpha(1 - \tau_t)}{(1 - \chi)\tau_t + \chi - \alpha} k_t^{-(1-\alpha)} \\ &\quad - \frac{c_t^{1-\frac{1}{\chi}}(1 - \tau_t)^{\frac{1}{\chi}}(1 - \alpha)^{\frac{1}{\chi}}((1 - \chi)\tau_t + \chi)}{(1 - \chi)\tau_t + \chi - \alpha} k_t^{\frac{\alpha}{\chi}-1} \\ &\quad + \frac{\rho(1 - \tau_t)}{(1 - \chi)\tau_t + \chi - \alpha},\end{aligned}$$

and

$$\begin{aligned}\frac{\dot{\tau}_t}{\tau_t} &= \frac{1 - \tau_t}{(1 - \chi)\tau_t + \chi} \left[\alpha \frac{(\chi + (1 - \chi)\tau_t)((\alpha - \chi) - (1 - \chi)(1 - \alpha)\tau_t)}{(1 - \chi)\tau_t + \chi - \alpha} k_t^{\alpha-1} \right. \\ &\quad - \frac{\alpha(1 - \chi)c_t^{1-\frac{1}{\chi}}(1 - \tau_t)^{\frac{1}{\chi}}(1 - \alpha)^{\frac{1}{\chi}}((1 - \chi)\tau_t + \chi)}{(1 - \chi)\tau_t + \chi - \alpha} k_t^{\frac{\alpha}{\chi}-1} \\ &\quad \left. + \frac{\rho(1 - \alpha)((1 - \chi)\tau_t + \chi)}{(1 - \chi)\tau_t + \chi - \alpha} \right].\end{aligned}$$

Moreover, we introduce new variables: $\log k_t = \lambda_t$, $\log c_t = \delta_t$ and $\log \tau_t = \eta_t$.

By this transformation, our dynamic system consists of the following three simultaneous differential equations in relation to δ , λ and η :

$$\dot{\delta}_t = \alpha e^{-(1-\alpha)\lambda_t} - \rho,$$

$$\begin{aligned}\dot{\lambda}_t &= \frac{(\alpha + (1 - \alpha)(1 - e^{\eta_t}))((1 - \chi)e^{\eta_t} + \chi) - \alpha(1 - e^{\eta_t})}{(1 - \chi)e^{\eta_t} + \chi - \alpha} e^{(\alpha-1)\lambda_t} \\ &\quad - \frac{e^{\frac{\chi-1}{\chi}\delta_t}(1 - e^{\eta_t})^{\frac{1}{\chi}}(1 - \alpha)^{\frac{1}{\chi}}((1 - \chi)e^{\eta_t} + \chi)}{(1 - \chi)e^{\eta_t} + \chi - \alpha} e^{(\frac{\alpha}{\chi}-1)\lambda_t} \\ &\quad + \frac{\rho(1 - e^{\eta_t})}{(1 - \chi)e^{\eta_t} + \chi - \alpha},\end{aligned}$$

and

$$\begin{aligned}\dot{\eta}_t &= \frac{1 - e^{\eta_t}}{(1 - \chi)e^{\eta_t} + \chi} \left[\alpha \frac{(\chi + (1 - \chi)e^{\eta_t})((\alpha - \chi) - (1 - \chi)(1 - \alpha)e^{\eta_t})}{(1 - \chi)e^{\eta_t} + \chi - \alpha} e^{(\alpha-1)\lambda_t} \right. \\ &\quad - \frac{\alpha(1 - \alpha)^{\frac{1}{\chi}}(1 - \chi)e^{\frac{\chi-1}{\chi}\delta_t}(1 - e^{\eta_t})^{\frac{1}{\chi}}((1 - \chi)e^{\eta_t} + \chi)}{(1 - \chi)e^{\eta_t} + \chi - \alpha} e^{(\frac{\alpha}{\chi}-1)\lambda_t} \\ &\quad \left. + \frac{\rho(1 - \alpha)((1 - \chi)e^{\eta_t} + \chi)}{(1 - \chi)e^{\eta_t} + \chi - \alpha} \right].\end{aligned}\tag{1}$$

It is evident that in the above dynamic system, there are two steady states. Moreover, two values in relation to τ^* are obtained by solving the following equation with an argument x ;

$$x(1 - B(1 - x)^{1-\chi}) + \frac{\alpha}{1 - \alpha} = 0.$$

Here, $x = 1 - \tau^*$ and $B = \left(\frac{w^*}{G}\right)^{1-\chi}$. The asterisk indicates the steady state.

3 Linearization

Based on what we have analyzed so far, we can express our dynamic system as follows:

$$\begin{cases} \dot{\delta}_t = f^1(\lambda_t), \\ \dot{\lambda}_t = f^2(\delta_t, \lambda_t, \eta_t), \\ \dot{\eta}_t = f^3(\delta_t, \lambda_t, \eta_t). \end{cases} \quad (2)$$

We linearize (2) around the steady state $(\delta^*, \lambda^*, \eta^*)$, we confirmed in the first paper. Thus, we obtain the following simpler expressions:

$$\begin{cases} \dot{\mathbf{z}} = A\mathbf{z}, \\ \mathbf{z} = [z_1, z_2, z_3]' = [\delta_t - \delta^*, \lambda_t - \lambda^*, \eta_t - \eta^*]'. \end{cases} \quad (3)$$

Here, the notation $'$ shows a transposition of a vector. A is defined as

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Moreover, we define elements in A as follows:

$$\begin{aligned} a_{12} &= \frac{\partial f^1}{\partial \lambda}, & a_{21} &= \frac{\partial f^2}{\partial \delta}, & a_{22} &= \frac{\partial f^2}{\partial \lambda}, & a_{23} &= \frac{\partial f^2}{\partial \eta} \\ a_{31} &= \frac{\partial f^3}{\partial \delta}, & a_{32} &= \frac{\partial f^3}{\partial \lambda}, & a_{33} &= \frac{\partial f^3}{\partial \eta}. \end{aligned}$$

In addition, let us return to (1), which shows that:

$$\eta_t = \frac{1 - e^{\eta t}}{(1 - \chi)e^{\eta t} + \chi} [\alpha(1 - \chi)f^2(\delta_t, \lambda_t, \eta_t) - f^1(\lambda_t)].$$

From the above equation, we obtain the following:

$$\begin{cases} \frac{\partial f^3}{\partial \delta_t} = \frac{\alpha(1-\chi)x}{(\chi-1)x+1} \frac{\partial f^2}{\partial \delta_t}, \\ \frac{\partial f^3}{\partial \lambda_t} = \frac{x}{(\chi-1)x+1} \left[\alpha(1-\chi) \frac{\partial f^2}{\partial \lambda_t} - \frac{\partial f^1}{\partial \lambda_t} \right], \\ \frac{\partial f^3}{\partial \eta_t} = \frac{\alpha(1-\chi)x}{(\chi-1)x+1} \frac{\partial f^2}{\partial \eta_t}. \end{cases} \quad (4)$$

Here, we can express (4) in terms of a_{ij} , Δ and Δ' , as $a_{31} = \Delta a_{21}$, $a_{32} = \Delta a_{22} - \Delta' a_{12}$, and $a_{33} = \Delta a_{23}$.

Additionally, Δ and Δ' are respectively defined as $\Delta = \frac{\alpha(1-\chi)x}{(\chi-1)x+1}$ and $\Delta' = \frac{x}{(\chi-1)x+1}$. Of course, in this context, x is a constant in terms of η^* , which shows the value of η in a steady state. In other words, $x = 1 - e^{\eta^*}$.

Thus, we can express A as

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ \Delta a_{21} & \Delta a_{22} - \Delta' a_{12} & \Delta a_{23} \end{pmatrix}. \quad (5)$$

This matrix controls the movement patterns in our fundamental system.

4 Movement patterns

At this stage, we need to detect both eigenvalues and corresponding eigenvectors in A , because we want to investigate how our economy moves under a fundamental economic condition.

4.1 Eigenvalues and eigenvectors

First, we seek eigenvalues of the Jacobian matrix with 3×3 elements in (5). We denote them as θ . After some calculations, we obtain a characteristic polynomial as

$$\phi(\theta) = |\theta I - A| = -\theta[\theta^2 - (a_{22} + \Delta a_{23})\theta + a_{12}(\Delta' a_{23} - a_{21})].$$

Based on the above polynomial, we obtain three eigenvalues as follows:

$$\begin{cases} \theta_1 = 0, \\ \theta_2 = \frac{1}{2}(a_{22} + \Delta a_{23} - \sqrt{(a_{22} + \Delta a_{23})^2 + 4a_{12}(a_{21} - \Delta' a_{23})}), \\ \theta_3 = \frac{1}{2}(a_{22} + \Delta a_{23} + \sqrt{(a_{22} + \Delta a_{23})^2 + 4a_{12}(a_{21} - \Delta' a_{23})}). \end{cases}$$

Second, we seek corresponding eigenvectors.

- (1) As for $\theta_1 = 0$,
the eigenvector can be described as

$$[h_1^1, h_1^2, h_1^3]' = \left[-\frac{a_{23}}{a_{21}}, 0, 1 \right]'$$

- (2) As for θ_i , the following holds⁵ :

$$\begin{pmatrix} -\theta_i & a_{22} & 0 \\ a_{21} & a_{22} - \theta_i & a_{23} \\ \Delta a_{21} & \Delta a_{22} - \Delta' a_{12} & \Delta a_{23} - \theta_i \end{pmatrix} \begin{pmatrix} h_i^1 \\ h_i^2 \\ h_i^3 \end{pmatrix} = 0.$$

Additionally, $i = 2, 3$.

Manipulating the above formula, we eventually obtain the following eigenvectors required:

$$[h_i^1, h_i^2, h_i^3]' = \left[\frac{a_{12}}{\Delta\theta_i - a_{12}\Delta'}, \frac{\theta_i}{\Delta\theta_i - a_{12}\Delta'}, 1 \right]'$$

where we assume $\theta_2 \neq \theta_3$.

We need to define the following matrix, P , which consists of the eigenvectors derived in the calculations above:

$$P = \begin{pmatrix} -\frac{a_{23}}{a_{21}} & \frac{a_{12}}{\Delta\theta_2 - a_{12}\Delta'} & \frac{a_{12}}{\Delta\theta_3 - a_{12}\Delta'} \\ 0 & \frac{\theta_2}{\Delta\theta_2 - a_{12}\Delta'} & \frac{\theta_3}{\Delta\theta_3 - a_{12}\Delta'} \\ 1 & 1 & 1 \end{pmatrix}. \quad (6)$$

⁵We note the elements in the eigenvectors as h_i^j , below. Here, i indicates the number identifying eigenvalues, and j the number showing places within the corresponding eigenvector in A , respectively.

Now, we can derive the inverse of P as follows:

$$P^{-1} = \begin{pmatrix} \frac{a_{21} \Delta'}{R} & \frac{-a_{21} \Delta}{R} & \frac{a_{21}}{R} \\ \frac{a_{21} \theta_3 (-\Delta \theta_2 + a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)R} & \frac{(\Delta \theta_2 - a_{12} \Delta')(\Delta a_{23} \theta_3 + a_{12} R)}{a_{12}(\theta_2 - \theta_3)R} & \frac{a_{23} \theta_3 (-\Delta \theta_2 + a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)R} \\ \frac{a_{21} \theta_2 (\Delta \theta_3 - a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)R} & -\frac{a_{12}(\theta_2 - \theta_3)R}{(\Delta \theta_3 - a_{12} \Delta')(\Delta a_{23} \theta_2 + a_{12} R)} & \frac{a_{23} \theta_2 (\Delta \theta_3 - a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)R} \end{pmatrix},$$

where $R = a_{21} - a_{23} \Delta'$.⁶

4.2 Transformation of the original system

Based on the arguments so far, we introduce a Jordan canonical form of A .

Multiplying both sides in (3) by P^{-1} , which is defined from (6), we obtain the following relationship, $P^{-1} \dot{\mathbf{z}} = P^{-1} A \mathbf{z}$, and Therefore $\mathbf{y} = P^{-1} \mathbf{z}$.⁷ Thus, the system with regard to \mathbf{y} is established as

$$\dot{\mathbf{y}} = P^{-1} A P \mathbf{y} = J(A) \mathbf{y}. \quad (7)$$

Since $J(A)$ is obviously a Jordan canonical form of A , the movements with regard to \mathbf{z} in the original system are in the same vein as those with regard to \mathbf{y} . So, we investigate system (7) below.

Under this assumption, when we denote (7) in more detail, the following hold:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \quad (8)$$

Solving (8), we have solutions as follows:

$$\begin{cases} y_1 = b \text{ (constant)}, \\ y_2 = y_2(0) e^{\theta_2 t}, \\ y_3 = y_3(0) e^{\theta_3 t}. \end{cases} \quad (9)$$

Here, $y_2(0)$ and $y_3(0)$ show initial conditions concerning y_2 and y_3 , respectively.

⁶Note that P^{-1} exists and is regular, because we assume that $\theta_2 \neq \theta_3$ and $a_{21} - a_{23} \Delta' \neq 0$.

⁷ \mathbf{y} indicates a vector with three rows and one column, and is expressed as $\mathbf{y} = [y_1, y_2, y_3]'$.

Before we analyze the whole system described in (9), first of all, we need to prove $y_1(t) = 0$. Let us return to the definition $\mathbf{y} = P^{-1}\mathbf{z}$. Expressing this relationship concretely, we obtain the following:

$$\left\{ \begin{array}{l} (a_{21} - a_{23} \Delta')y_1 = a_{21} \Delta' z_1 - \Delta a_{21}z_2 + a_{21}z_3, \\ (a_{21} - a_{23} \Delta')y_2 = \frac{-a_{21} \theta_3(\Delta \theta_2 - a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)} z_1 \\ + \frac{(\Delta \theta_2 - a_{12} \Delta')(\Delta a_{23}\theta_3 + a_{12}(a_{21} - a_{23} \Delta'))}{a_{12}(\theta_2 - \theta_3)} z_2 - \frac{a_{23} \theta_3(\Delta \theta_2 - a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)} z_3, \\ (a_{21} - a_{23} \Delta')y_3 = \frac{a_{21} \theta_2(\Delta \theta_3 - a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)} z_1 \\ - \frac{(\Delta \theta_3 - a_{12} \Delta')(\Delta a_{23}\theta_2 + a_{12}(a_{21} - a_{23} \Delta'))}{a_{12}(\theta_2 - \theta_3)} z_2 + \frac{a_{23} \theta_2(\Delta \theta_3 - a_{12} \Delta')}{a_{12}(\theta_2 - \theta_3)} z_3. \end{array} \right.$$

We assumed the transversality condition, which satisfies an optimal condition with regard to household budgets. And this condition requires z_1 , z_2 and z_3 to converge to null at infinity in terms of time. Therefore, $y_1 = b = 0$ must hold. Note that $y_1(t)$ is constant.

In this situation, we can consider that the system is autonomously controlled by only two variables, y_2 and y_3 . This means that we can concretely focus on the following system alone:

$$\begin{pmatrix} \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_3 \end{pmatrix} \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}. \quad (10)$$

The characteristic polynomial from this system is obviously shown as

$$\theta^2 - (a_{22} + \Delta a_{23})\theta + a_{12}(\Delta' a_{23} - a_{21}).$$

Now, we need to explore our system more intensively.

Based on P , we can establish our system through linear transformations of $P\mathbf{y} = \mathbf{z}$, as follows:

$$\begin{aligned} P\mathbf{y} &= \begin{pmatrix} -\frac{a_{23}}{a_{21}} & \frac{a_{12}}{\Delta\theta_2 - a_{12}\Delta'} & \frac{a_{12}}{\Delta\theta_3 - a_{12}\Delta'} \\ 0 & \frac{\theta_2}{\Delta\theta_2 - a_{12}\Delta'} & \frac{\theta_3}{\Delta\theta_3 - a_{12}\Delta'} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{a_{12}}{\Delta\theta_2 - a_{12}\Delta'} y_2 + \frac{a_{12}}{\Delta\theta_3 - a_{12}\Delta'} y_3 \\ \frac{\theta_2}{\Delta\theta_2 - a_{12}\Delta'} y_2 + \frac{\theta_3}{\Delta\theta_3 - a_{12}\Delta'} y_3 \\ y_2 + y_3 \end{pmatrix} \\ &= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}. \end{aligned}$$

Thus, we obtain the following relationships:

$$\begin{cases} z_1 = \frac{a_{12}}{\Delta\theta_2 - a_{12}\Delta'} y_2 + \frac{a_{12}}{\Delta\theta_3 - a_{12}\Delta'} y_3, \\ z_2 = \frac{a_{12}}{\Delta\theta_2 - a_{12}\Delta'} y_2 + \frac{a_{12}}{\Delta\theta_3 - a_{12}\Delta'} y_3, \\ z_3 = y_2 + y_3. \end{cases}$$

Moreover, the following hold:⁸ $\Delta z_2 - \Delta' z_1 = y_2 + y_3$, $y_1 + y_2 + y_3 = z_3$, and $y_1 = 0$.

In short, this eventually means $z_3 = \Delta z_2 - \Delta' z_1$ in terms of \mathbf{z} . Thus, it becomes obvious that z_3 is a linear dependence, following z_1 and z_2 .

From what we have analyzed so far, we can say that when y_2 and y_3 are determined, z_1 and z_2 are correspondingly determined. Thus, z_3 is also determined. In this context, it is clear that movements by y_2 and y_3 control all the movements in our system.

5 Determination of initial conditions

In more detail, \mathbf{y} is determined by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \omega_1 \begin{pmatrix} h_1 \\ 0 \\ 0 \end{pmatrix} + \omega_2 \begin{pmatrix} 0 \\ h_2 \\ 0 \end{pmatrix} e^{\theta_2 t} + \omega_3 \begin{pmatrix} 0 \\ 0 \\ h_3 \end{pmatrix} e^{\theta_3 t}.$$

Here, of course, ω_i ($i = 1, 2, 3$) is a constant which is determined by initial conditions, $y_1(0)$, $y_2(0)$ and $y_3(0)$.

Now, let us more specifically investigate this determination with regard to \mathbf{y} .

First of all, since $y_1(t) = 0$, it is clear that $\omega_1 = 0$.

Second, let us deal with the determination concerning initial conditions of \mathbf{y} . Based on the preceding simultaneous equation system, we easily know the following:

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \omega_1 \begin{pmatrix} h_1 \\ 0 \\ 0 \end{pmatrix} + \omega_2 \begin{pmatrix} 0 \\ h_2 \\ 0 \end{pmatrix} + \omega_3 \begin{pmatrix} 0 \\ 0 \\ h_3 \end{pmatrix}.$$

⁸The first relationship is derived from $\mathbf{y} = P^{-1}\mathbf{z}$, which was defined.

Here, we should notice that initial conditions with regard to \mathbf{y} are determined by the original system. In other words, initial conditions with regard to \mathbf{z} determine those in \mathbf{y} . Given that $y_1 = 0$, we obtain the following as a relationship among initial conditions in the original system and those in \mathbf{y} :

$$\begin{cases} z_1(0) = \frac{a_{12}}{\Delta\theta_2 - a_{12}\Delta'} y_2(0) + \frac{a_{12}}{\Delta\theta_3 - a_{12}\Delta'} y_3(0), \\ z_2(0) = \frac{\theta_2}{\Delta\theta_2 - a_{12}\Delta'} y_2(0) + \frac{\theta_3}{\Delta\theta_3 - a_{12}\Delta'} y_3(0), \end{cases}$$

and

$$\begin{cases} z_3(0) = \Delta z_2(0) - \Delta' z_1(0), \\ z_3(0) = y_2(0) + y_3(0). \end{cases}$$

According to the above relationships, when $z_1(0)$ and $z_2(0)$ are given, $y_2(0)$ and $y_3(0)$ are determined, and, as a result, a whole system is finally determined.

On the other hand, solutions concerning the original system are given as

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \omega'_1 \begin{pmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \end{pmatrix} + \omega'_2 \begin{pmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{pmatrix} e^{\theta_2 t} + \omega'_3 \begin{pmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \end{pmatrix} e^{\theta_3 t}.$$

Similarly, ω'_i ($i = 1, 2, 3$) is a constant which is determined by initial conditions, $z_1(0)$, $z_2(0)$ and $z_3(0)$.

Therefore, arbitrary initial conditions satisfy the following formula:

$$\begin{pmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{pmatrix} = \omega'_1 \begin{pmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \end{pmatrix} + \omega'_2 \begin{pmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{pmatrix} + \omega'_3 \begin{pmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \end{pmatrix}.$$

Given $z_3(0) = \Delta z_2(0) - \Delta' z_1(0)$, we do not need to consider how $z_3(0)$ is to be determined. Therefore, here, we can explore specifically how $z_1(0)$ and $z_2(0)$ alone are determined. Thus, we consider the following:

$$\begin{pmatrix} \log \left(\frac{c_0}{c^*} \right) \\ \log \left(\frac{K^{**}}{H_0} \right) \\ \log \left(\frac{K^*}{H^*} \right) \end{pmatrix} = \omega'_2 \begin{pmatrix} h_1^{2*} \\ h_2^{2*} \end{pmatrix} + \omega'_3 \begin{pmatrix} h_1^{3*} \\ h_2^{3*} \end{pmatrix}. \quad (11)$$

It is evident that (11) can be eventually derived from the following relationships:

$$\begin{cases} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \omega'_2 \begin{pmatrix} \xi_2 h_1^1 + h_1^2 \\ \xi_2 h_2^1 + h_2^2 \end{pmatrix} + \omega'_3 \begin{pmatrix} \xi_3 h_1^1 + h_1^3 \\ \xi_3 h_2^1 + h_2^3 \end{pmatrix}, \\ z_3(0) = \omega'_2(\xi_2 h_3^1 + h_3^2) + \omega'_3(\xi_3 h_3^1 + h_3^3). \end{cases}$$

Additionally, $h_1^{2*} = \xi_2 h_1^1 + h_1^2$, $h_2^{2*} = \xi_2 h_2^1 + h_2^2$, $h_1^{3*} = \xi_3 h_1^1 + h_1^3$ and $h_2^{3*} = \xi_3 h_2^1 + h_2^3$.

And here, we define

$$\begin{cases} \xi_2 = -\frac{\Delta h_2^2 - \Delta' h_1^2 - h_3^2}{\Delta h_2^1 - \Delta' h_1^1 - h_3^1}, \\ \xi_3 = -\frac{\Delta h_2^3 - \Delta' h_1^3 - h_3^3}{\Delta h_2^1 - \Delta' h_1^1 - h_3^1}. \end{cases}$$

At this stage, first, we assume that the real part of eigenvalues concerning eigenvector $(h_1^2, h_2^2)'$ is negative, while that concerning eigenvector $(h_1^3, h_2^3)'$ is positive.

Our system consists only of a predetermined state, variable K^{**} and three other adjoint variables such as H , c and τ .

In general, when the state variable is given, ω'_2 is determined. Then initial conditions concerning the adjoint variables are determined, under ω'_2 . This mechanism shows the role of a predetermined variable, K^{**} , upon the determination of initial conditions with regard to adjoint variables.

However, since we deal with K/H as one variable, which is a ratio between a state variable and an adjoint one, our story is a bit complicated.

Here, let us introduce the following:⁹

$$\begin{cases} \log\left(\frac{\tau_0}{\tau^*}\right) = \Delta \log\left[\frac{\left(\frac{K^{**}}{H_0}\right)}{\left(\frac{K^*}{H^*}\right)}\right] - \Delta' \log\left(\frac{c_0}{c^*}\right), \\ H_0 = \left[\frac{c_0}{(1-\tau_0)w_0}\right]^{\frac{1}{\chi}}. \end{cases} \quad (12)$$

5.1 Initial conditions leading to a saddle path

With this preparation, we can investigate how ω'_2 and initial conditions concerning adjoint variables are determined.

K^{**} is of course predetermined. If H_0 is arbitrary, ω'_2 can be solved by the second equation in (11). Let us denote this solution as ω'_2^* . Then, c_0 can be solved by the first equation in (11). Of course, in this context, $\omega'_3 = 0$.

However, it is only when H is arbitrary that this logic holds. In order to decide H_0 , we need one more relationship. That is why the second relationship in (12) is introduced, which shows the initial supply of a labor force.

From what we have analyzed so far, the initial conditions of H , c , τ and a constant ω'_2^* are decided by (11) and (12). In this situation, it is obvious that $\omega'_3 = 0$. In this way, since $z_1(0)$ and $z_2(0)$ can be determined uniquely, $y_2(0)$ and $y_3(0)$ are also determined uniquely. This is a determination of initial conditions leading to a saddle path.

This logic is crucial, so we need to confirm in more detail that ω'_2^* is determined by K^{**} alone.

From the second equation in (11), ω'_2^* is transformed to

$$\omega'_2^* = \frac{1}{h_2^{2*}} \log\left(\frac{\left(\frac{K^{**}}{H_0}\right)}{k^*}\right). \quad (13)$$

⁹The first equation in (12) comes from $z_3(0) = \Delta z_2(0) - \Delta' z_1(0)$. Additionally, $z_1 = \log\left(\frac{c}{c^*}\right)$, $z_2 = \log\left(\frac{k}{k^*}\right)$ and $z_3 = \log\left(\frac{\tau}{\tau^*}\right)$. Notice that w is a function determined by k .

On the other hand, from the first equation in (11), c_0 is determined as

$$\log\left(\frac{c_0}{c^*}\right) = \omega_2' h_1^{2*} = \left(\frac{h_1^{2*}}{h_2^{2*}}\right) \log\left(\frac{K^{**}}{H_0}\right). \quad (14)$$

Substituting the above two relationships into $z_3(0) = \Delta z_2(0) - \Delta' z_1(0)$, we obtain a relationship concerning τ_0 as

$$\log\left(\frac{\tau_0}{\tau^*}\right) = \left[\Delta - \Delta' \left(\frac{h_1^{2*}}{h_2^{2*}}\right)\right] \log\left(\frac{K^{**}}{H_0}\right). \quad (15)$$

Here, remember the supply function with regard to the labor force, which is denoted in (12). Transforming and arranging this function, we eventually obtain

$$\chi \log H_0 = \log c_0 - \log(1 - \tau_0) - \log w_0. \quad (16)$$

Substituting (14), (15) and $w_0 = (1 - \alpha)k_0^\alpha$ into (16) and arranging them, the following holds:

$$\begin{aligned} \chi \log H_0 &= \left(\frac{h_1^{2*}}{h_2^{2*}}\right) \log\left(\frac{K^{**}}{H_0}\right) \\ &+ \log c^* - \log \left[1 - \tau^* \exp \left[\left(\Delta - \Delta' \left(\frac{h_1^{2*}}{h_2^{2*}} \right) \right) \log \frac{K^{**}}{H_0} \right] \right] \\ &- \log(1 - \alpha) - \alpha \log\left(\frac{K^{**}}{H_0}\right). \end{aligned} \quad (17)$$

Based on the above relationship, we can know that if K^{**} is predetermined, H_0 can be determined by K^{**} alone. We can describe this as $H_0 = \psi(K^{**})$. Thus, $\omega_2'^*$ can be expressed as

$$\omega_2'^* = \frac{1}{h_2^{2*}} \log\left(\frac{K^{**}}{\psi(K^{**})}\right),$$

which shows that $\omega_2'^*$ is determined by K^{**} alone.

Thus a saddle path exists in this context.

5.2 Initial conditions leading to indeterminacy paths

Second, we assume that the real part of the eigenvalue concerning eigenvector $(h_1^3, h_2^3)'$ is also negative. In this case, ω_3' as well as H , c , τ and ω_2^* need to be determined by the previous four equations, that is, by (11) and (12).

However, in this case, since the number of equations is short of that of the variables, the values of the variables are not uniquely determined. In other words, H_0 , c_0 , τ_0 , ω_2^* , ω_3^* are arbitrary. Thus, $z_1(0)$ and $z_2(0)$ are indeterminate, and so are $y_2(0)$ and $y_3(0)$. Therefore, the paths which we are interested in are indeterminate. We can say that there can be no price mechanism, which means there is no identified path.

6 Indeterminacy paths and a saddle path

Based on the arguments in the last section, we have established the possibility of two different movement patterns, namely indeterminacy paths and a saddle path. And in the previous paper, Takata (2013), we established the existence of two steady states in our system.

In this section, we intend to demonstrate that one path toward one steady state is a saddle path, while other paths toward another steady state are indeterminate paths.

6.1 Conditions for indeterminacy

We have established that our system movements are controlled by (10). In terms of this system, we explore conditions under which indeterminacy occurs. In our system, the following conditions are necessary and sufficient for indeterminacy:

$$\theta_1 + \theta_2 = a_{22} + \Delta a_{23} < 0 \quad (18)$$

and

$$\theta_1 \theta_2 = a_{12} (\Delta^* a_{23} - a_{21}) > 0. \quad (19)$$

Below, we show that the constants in the right side in the above relationships are respectively expressed by a function of τ^* , except for a_{12} .¹⁰ We obtain

¹⁰In the analyses below, the asterisk with τ is omitted for simplicity.

the following:

$$\left\{ \begin{array}{l}
\Delta(\tau) = \frac{\alpha(1-\tau)(1-\chi)}{1+(1-\tau)(-1+\chi)}, \\
\Delta^*(\tau) = \frac{1-\tau}{1+(1-\tau)(-1+\chi)}, \\
a_{12} = (-1+\alpha)\rho, \\
a_{21}(\tau) = \frac{k^{-1+\frac{\alpha}{x}}w^{-\frac{-1+\chi}{x}}(1-\alpha)^{\frac{1}{x}}\tau^2(-1+\chi)^2}{(\alpha-\chi)\chi+\tau(-1+\chi)\chi} \\
+ \frac{(1+\kappa)k^{-1+\frac{\alpha}{x}}w^{-\frac{-1+\chi}{x}}(1-\alpha)^{\frac{1}{x}}(-1+\chi)\chi}{(\alpha-\chi)\chi+\tau(-1+\chi)\chi} \\
+ \frac{k^{-1+\frac{\alpha}{x}}w^{-\frac{-1+\chi}{x}}(1-\alpha)^{\frac{1}{x}}\tau(-1+\chi)(1+\kappa-2\chi-\kappa\chi)}{(\alpha-\chi)\chi+\tau(-1+\chi)\chi}, \\
a_{22}(\tau) = \frac{\frac{1}{k}w^{-1}\chi((1+\kappa)k^\alpha\chi w(1-\alpha)^{\frac{1}{x}}+k^\alpha w^{\frac{1}{x}}(-1+\alpha))(-\alpha+\chi)}{(-\alpha+\tau(1-\chi)+\chi)} \\
- \frac{\frac{1}{k\chi}w^{-1}\chi\tau^2(-1+\chi)(k^\alpha\chi w(1-\alpha)^{\frac{1}{x}}(\alpha-\chi)+k^\alpha w^{\frac{1}{x}}(-1+\alpha)^2\chi)}{(-\alpha+\tau(1-\chi)+\chi)} \\
+ \frac{\frac{1}{k\chi}w^{-\frac{1}{x}}\tau(k^\alpha\chi w(1-\alpha)^{\frac{1}{x}}(\alpha-\chi)(-1+\kappa(-1+\chi)+2\chi))}{(-\alpha+\tau(1-\chi)+\chi)} \\
+ \frac{\frac{1}{k\chi}w^{-\frac{1}{x}}\tau k^\alpha w^{\frac{1}{x}}(-1+\alpha)\chi(1+\alpha-2\chi+\alpha\chi)}{(-\alpha+\tau(1-\chi)+\chi)}, \\
a_{23}(\tau) = [(\tau\frac{1}{k}(w^{-1/x}(1-\tau)^{-\frac{1}{(1-x)x}}(1+\kappa-\tau)^{-\frac{1}{1-x}} \\
\times(-k^{\alpha/x}w(1-\alpha)^{\frac{1}{x}}(1-\tau)^{\frac{1}{(1-x)x}}(1+\kappa-\tau)^{-\frac{x}{1-x}} \\
\times(-1+\alpha-(1-\tau)^2(-1+\chi)^2+(1-\tau)(-1+\chi)(-2+\alpha+\alpha\chi)) \\
+(-1+\alpha)\rho(1-\tau)\chi k(w^{\frac{1}{x}}(1-\tau)^{\frac{1}{(1-x)x}}(1+\kappa-\tau)^{-\frac{1}{1-x}}) \\
+(1-\tau)(-1+\alpha(3+\alpha(-3+\chi)))-2(-1+\alpha)^2(1-\tau)(-1+\chi) \\
+(-1+\alpha)(1-\tau)^2(-1+\chi)^2)\chi k^\alpha(w^{\frac{1}{x}} \\
\times(1-\tau)^{\frac{1}{(1-x)x}}(1+\kappa-\tau)^{-\frac{1}{1-x}}))] / ((1-\tau)\chi(\alpha-\tau-(1-\tau)\chi)^2).
\end{array} \right. \quad (20)$$

In (20), constants are respectively shown as follows: $k = (\frac{\alpha}{\rho})^{\frac{1}{1-\alpha}}$, $w = (1-\alpha)k^\alpha$, and $\kappa = \frac{\alpha}{1-\alpha}$, which refers to a ratio of capital in output to that of labor in output.

Based on the above relationships, we can examine what conditions cause (18) and (19). Since we can consider from (20) that both (18) and (19) respectively are a function of τ , we focus on how the shapes of functions

move according to changes in τ . In this context, τ is determined by the two intersections, where government spending equals tax revenue. As a result, two tax rates exist, one larger, another smaller, corresponding to tax spending given¹¹. It is only with the two τ s determined in the above way that we consider (18) and (19). Naturally, in this situation, we assume that the tax spending given is changeable.

With the above preparations, we can obtain expressions concerning $\theta_1 + \theta_2$ and $\theta_1\theta_2$, which consist only of fundamental parameters such as ρ , α and χ , as follows¹² :

$$t(\tau) = \theta_1 + \theta_2 = \frac{\varphi^1(\tau)}{((\alpha - \chi) - (1 - \chi)\tau)^2(1 - \chi)\tau + \chi}, \quad (21)$$

$$d(\tau) = \theta_1\theta_2 = \frac{\varphi^2(\tau)}{((\alpha - \chi) - (1 - \chi)\tau)^2((1 - \chi)\tau + \chi)}. \quad (22)$$

Here, φ^i , $i = 1, 2$, are functions satisfying (18), (19) and (20).

In light of (20), it is obvious that both $\Delta(\tau)$ and $\Delta^*(\tau)$ respectively have a common asymptote, $\tau = \frac{-\chi}{1-\chi}$, and that $a_{21}(\tau)$, $a_{22}(\tau)$ and $a_{23}(\tau)$ similarly have a common asymptote, $\tau = \frac{\alpha-\chi}{1-\chi}$. In other words, since we assumed $\tau \neq \frac{-\chi}{1-\chi}$ and $\tau \neq \frac{\alpha-\chi}{1-\chi}$ in the previous paper, these assumptions are coincident with the appearance of these asymptotes in this context.

In general, an asymptote can cause a curve to shift to discontinuity. Here, since we consider both $\theta_1 + \theta_2$ and $\theta_1\theta_2$ as a curve with an independent variable τ , we can easily speculate that at these points where two asymptotes hold, the sign of $\theta_1 + \theta_2$ and $\theta_1\theta_2$ can change. Moreover, the above curves have common asymptotes. This implies a possibility that at these common points in terms of τ , indeterminacy begins to occur. However, between the two points, at $\tau = \frac{-\chi}{1-\chi}$, the value of $\theta_1 + \theta_2$ and $\theta_1\theta_2$ together expands to infinity in terms of a complex variable¹³. Therefore, the asymptote does not affect the sign of $\theta_1 + \theta_2$ and $\theta_1\theta_2$, unless $\tau = \frac{\alpha-\chi}{1-\chi}$. That is, at $\tau = \frac{\alpha-\chi}{1-\chi}$ our economic system begins to show indeterminacy if some conditions in relation to fundamental parameters such as α and χ are satisfied. Strictly speaking, ρ can also influence the sign. However, we omit this possibility, because we

¹¹See Takata (2013).

¹²See details in the appendix file which describes a program inducing (21) and (22).

¹³See detailed explanations in the appendix file

focus on labor supply and ρ is not included in our labor supply function. In order to verify the above inference, let us visualize $\theta_1 + \theta_2$ and $\theta_1\theta_2$ in a figure. Suppose $\rho = 0.05$, $\alpha = 0.5$ and $\chi = -0.1$. Then, we obtain figure 1, where. τ^* is a tax rate corresponding to the maximum tax revenue.

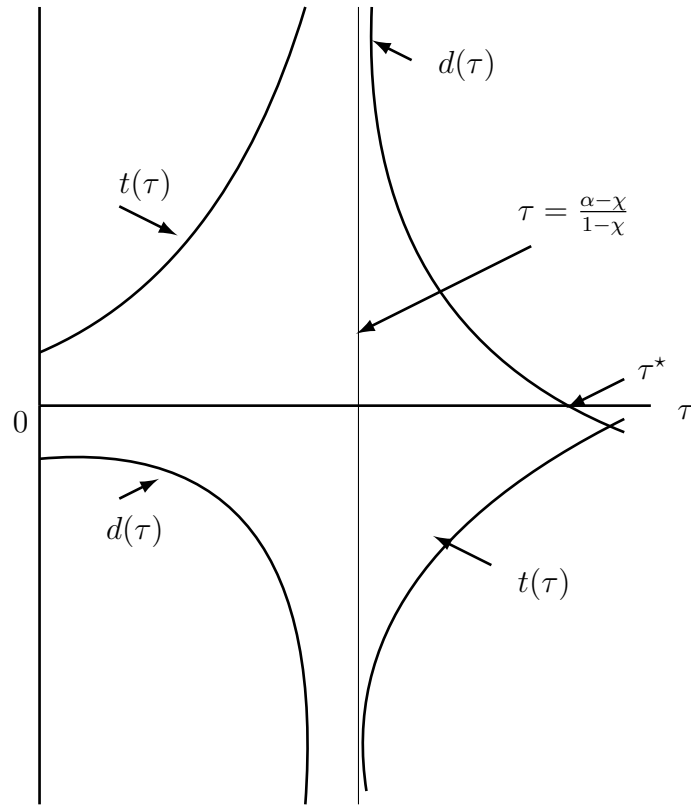


Figure 1: Given $\frac{\alpha - \chi}{1 - \chi} < \tau < \tau^*$, indeterminacy exists.

We can see that at $\tau = \frac{\alpha-\chi}{1-\chi}$, the sign of $\theta_1 + \theta_2$ changes from positive to negative, and that of $\theta_1\theta_2$ changes from negative to positive, as τ moves from null to 1. And this shows that the movement pattern in our economy changes from a saddle path to indeterminacy, as τ moves from null to 1. Of course, given $0 < \tau < \frac{\alpha-\chi}{1-\chi}$ and $\tau > \tau^*$, saddle paths exist, while given $\frac{\alpha-\chi}{1-\chi} < \tau < \tau^*$, indeterminacy exists..

At this stage, let us investigate indeterminacy from an economic viewpoint more intensively. Naturally, we should mainly focus on the labor supply. A representative household decides how much time it offers anticipating future income tax rates, and the tax rates are finally decided in order to observe the balanced budget principle.

First, suppose that household anticipates a rise in the tax rates in the future. This anticipation causes it to reduce the labor supply in the future, which causes a reduction in the anticipated rate of return on capital, because of a rise in the ratio of capital to labor. And this reduction causes an increase in current consumption and a decrease in current labor supply.

Second, the decrease in current labor supply leads to a decrease in the current tax base, namely labor income tax. This makes government decide to impose higher tax rates in order to recover a balanced budget. In other words, the anticipation concerning a tax rise in the future can cause current high tax rates. Anticipation can be self-fulfilling. This is an interpretation in relation to indeterminacy.

However, this mechanism does not work unconditionally.

First, this tax hike might not cause an increase in tax revenue. Tax revenue curve, looks like a bell curve, which implies that there is some range in which high tax rates cause a reduction in tax revenue. We know that this occurs given a tax rate range over that corresponding to maximum revenue, that is, τ^* . Therefore, τ^* is the largest tax rate involving indeterminacy. Moreover, strictly speaking, we can not say whether τ^* is included in that range or not¹⁴.

Second, for a self-fulfilling situation to occur, the current increase in tax rates corresponding to the expected increase in future taxes need not be large. Even in a range of comparatively low tax rates, this phenomenon is likely to occur¹⁵. This is why $\frac{\alpha-\chi}{1-\chi}$ is considered to be the minimum range of

¹⁴Our model can not deal with this situation. See the Appendix.

¹⁵The same statement is made in Anagnostopoulous and Giannitsarou (2012).

tax rates.

6.2 An upper limitation

Based on the above analyses, it is obvious that for indeterminacy, $\frac{\alpha-\chi}{1-\chi}$ has to be less than τ^* . Since

$$\tau^* = \frac{\chi(2-\alpha) - 2 + \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1-\alpha)(\chi-1)},$$

the following has to be satisfied:

$$\frac{\chi(2-\alpha) - 2 + \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1-\alpha)(\chi-1)} > \frac{\alpha-\chi}{1-\chi}.$$

Calculating the above inequality, we obtain

$$0 > \chi > \frac{\alpha^3 - \alpha^2 + 2\alpha - 1}{\alpha^2}. \quad (23)$$

The above relationship shows that if a set $\{ \alpha, \chi \}$ satisfying (23) can exist, indeterminacy does. Additionally, we can consider that the right side in the above inequality is a function with an argument α . The function has properties such as an asymptote approaching negative infinity as α approaches null, increasing monotonously and being negative for $0 < \alpha < 1$. In this situation, since χ is negative, α has an upper limitation, approximately to 0.5698. We can easily verify this by solving the equation $\alpha^3 - \alpha^2 + 2\alpha - 1 = 0$, which corresponds to $\chi = 0$.

In general, if an elasticity of labor supply with respect to wages, $-1/\chi$, is given, there is an upper limitation concerning α , which is shown by (23), in order for indeterminacy to occur. The expression *Upper limitation concerning α* can be replaced by *concerning the ratio of capital in output to that of labor in output*. This explanation might be clearer than the one above from an economic viewpoint.

(23) can be transformed into the following

$$e = -\frac{1}{\chi} > \frac{-\alpha^2}{\alpha^3 - \alpha^2 + 2\alpha - 1} = \frac{\kappa^2(1+\kappa)}{1+\kappa-\kappa^3}. \quad (24)$$

Here, $\kappa = \frac{\alpha}{1-\alpha}$, which shows the ratio of capital in output to that of labor in output. In the above relationship, a function with an argument κ has

properties such as an asymptote at $\kappa = 1.3247$, increasing monotonously in regard with κ and being positive. Based on these properties, we can say that for a certain degree of the elasticity there is an upper limit concerning the ratio. And the maximum ratio in indeterminacy, which is determined by a fundamental parameter, α , is approximately 1.3247.

Hence we have a theorem:

Theorem 1 *When indeterminacy occurs, the ratio of capital in output to that of labor in output, κ , must be less than 1.324 approximately.*

7 Conclusions

We have analyzed how our economy under a balanced budget rule moves.

In the previous paper, we postulated a utility function which features increasing marginal disutility with regard to labor supply, that is, a divisible one, while Schmitt-Grohé and Uribe and their followers such as Anagnostopoulos and Giannitsarou posit indivisible labor. Due to our postulation, we can specifically deduce a labor supply function, which can be affected by α and χ , and it becomes obvious that the shape of the tax revenue function is like a bell curve, with an independent argument regarding tax rates at steady states. With the help of global analysis, we demonstrate that this causes the existence of two steady states on the assumption of the balanced budget rule.

Moreover, we demonstrate that in our economy, there is a maximum tax revenue. We have also demonstrated that at this maximum tax revenue, there is a double steady state.

Thus, in this paper, we conclude that with an evaluation of the linear approximation system in terms of a tax rate range, for a certain range of τ^* , our economy exhibits indeterminacy leading toward a superior steady state. In this context, the low edge is $(\alpha - \chi)/(1 - \chi)$, and the upper one is the crucial rate, which corresponds to the maximum tax revenue. Moreover, in order for this to hold, the ratio of capital in output to that of labor in output has to be less than 1.3247. On the other hand, at a higher tax rate than in this crucial range, it can converge to an inferior steady state along a saddle path.

These phenomena are caused by the assumption of the balanced budget principle. In other words, the existence of two steady states is caused by this

principle. This is shown in our model by the fact that τ is not an independent variable, and that an eigenvalue from the characteristic polynomial $\phi(\theta)$ is null, which is deduced from Jacobian matrix A . Thus, τ plays a role as a kind of variable parameter. As a consequence, when G changes, τ^* can change, and the characteristic polynomial $\phi(\theta)$ can correspondingly shift. Then, since the eigenvalues can change, two movement patterns appear, namely a saddle path and indeterminate paths.

In this situation, the government can have more freedom with regard to tax rate policies. Of course, since it is desirable to achieve higher performance, we should be careful about initial conditions, especially with regard to labor income tax rates in our economic system, in order to guide our economy to a steady state with higher performance. However, this theory is applicable only in the case of indeterminacy. Regarding the saddle path, since the initial condition of a state variable $K(0)$ determines initial conditions in relation to adjoined variables, there is no room for the government to intervene.

Appendix

On page 981, Schmitt-Grohé and Uribe (1997) offers the Laffer-Curve as

$$G = \left(\frac{s_h \delta}{s_i} \right) \tau K.$$

Here, $s_h = wH/F$ and $s_i = \delta K/F$, respectively. Then, they deduce a tax rate, τ^{**} , corresponding to a maximum revenue.¹⁶

In this situation,

$$s_h(\tau^{**})^2 - 2(1 - s_i)\tau^{**} + (1 - s_i) = 0 \quad (25)$$

is proved.

On the other hand, Schmitt-Grohé and Uribe (1997) establishes the following simultaneous differential equation system, which shows dynamics in their economy:

$$\begin{pmatrix} \dot{\lambda}_t \\ \dot{k}_t \end{pmatrix} = \begin{pmatrix} -(\rho + \delta) \frac{s_h(1 - \tau)}{s_k - \tau} & -(\rho + \delta) \frac{s_h \tau}{s_k - \tau} \\ \frac{\delta}{s_i} \left[\frac{s_h(1 - \tau)}{s_k - \tau} + s_c \right] & (\rho + \delta) \frac{1 - \tau}{s_k - \tau} - \delta \end{pmatrix} \begin{pmatrix} \lambda_t \\ k_t \end{pmatrix}. \quad (26)$$

¹⁶In their paper, it is denoted as τ^* , but we denote it as in (25) below.

(26) is established by linear approximation based on the original dynamic system. Naturally, indeterminacy should be dealt with based on a simultaneous differential equation system.

Necessarily, τ is determined by the Laffer-Curve, when G is given. Here, in (26), we call the matrix with 2×2 elements $J(\tau)$.

So, since in light of (25), it is proved that the value of $J(\tau^{**})$ is null, λ^* and k^* are not determined uniquely, based on an implicit function theorem. Of course, in terms of the linear approximation, the value of both k and λ at a steady state should be determined by the following simultaneous equation system:

$$J(\tau) \begin{pmatrix} \lambda^* \\ k^* \end{pmatrix} = 0.$$

And this eventually leads to indeterminacy with regard to τ^{**} , because (26) can be established on an assumption of the existence of steady states.

Therefore, the model in Schmitt-Grohé and Uribe (1997), which explores dynamics in terms of linear approximation, can not explain movement patterns at this steady state. This can be applicable to our model.

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Supplemental material

This material is a supplemental to the paper “Two Movement Patterns under the Balanced Budget Rule—Further Results.”

5 Determination of initial conditions

We again describe relationships concerning the determination of ω'_i as

$$\begin{cases} z_1(0) = \omega'_1 h_1^1 + \omega'_2 h_1^2 + \omega'_3 h_1^3, \\ z_2(0) = \omega'_1 h_2^1 + \omega'_2 h_2^2 + \omega'_3 h_2^3, \\ z_3(0) = \omega'_1 h_3^1 + \omega'_2 h_3^2 + \omega'_3 h_3^3. \end{cases}$$

Based on the above equations, $\Delta z_2(0) - \Delta' z_1(0)$ can be expressed as follows:

$$\Delta z_2(0) - \Delta' z_1(0) = (\Delta h_2^1 - \Delta' h_1^1) \omega'_1 + (\Delta h_2^2 - \Delta' h_1^2) \omega'_2 + (\Delta h_2^3 - \Delta' h_1^3) \omega'_3.$$

Then, given $\Delta z_2(0) - \Delta' z_1(0) = z_3(0)$, we can express ω'_1 in terms of ω'_2 and ω'_3 , as follows:

$$\omega'_1 = -\frac{\Delta h_2^2 - \Delta' h_1^2 - h_3^2}{\Delta h_2^1 - \Delta' h_1^1 - h_3^1} \omega'_2 - \frac{\Delta h_2^3 - \Delta' h_1^3 - h_3^3}{\Delta h_2^1 - \Delta' h_1^1 - h_3^1} \omega'_3,$$

which can be further described as

$$\omega'_1 = \xi_2 \omega'_2 + \xi_3 \omega'_3.$$

Additionally, we define

$$\begin{cases} \xi_2 = -\frac{\Delta h_2^2 - \Delta' h_1^2 - h_3^2}{\Delta h_2^1 - \Delta' h_1^1 - h_3^1}, \\ \xi_3 = -\frac{\Delta h_2^3 - \Delta' h_1^3 - h_3^3}{\Delta h_2^1 - \Delta' h_1^1 - h_3^1}. \end{cases}$$

In this situation, we obtain the following system, which can determine ω'_2 and ω'_3 without the third equation concerning $z_3(0)$:

$$\begin{pmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{pmatrix} = (\xi_2 \omega'_2 + \xi_3 \omega'_3) \begin{pmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \end{pmatrix} + \omega'_2 \begin{pmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{pmatrix} + \omega'_3 \begin{pmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \end{pmatrix}.$$

In order to confirm this, we transform the above relationship as

$$\begin{cases} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = \omega'_2 \begin{pmatrix} \xi_2 h_1^1 + h_1^2 \\ \xi_2 h_2^1 + h_2^2 \end{pmatrix} + \omega'_3 \begin{pmatrix} \xi_3 h_1^1 + h_1^3 \\ \xi_3 h_2^1 + h_2^3 \end{pmatrix}, \\ z_3(0) = \omega'_2(\xi_2 h_3^1 + h_3^2) + \omega'_3(\xi_3 h_3^1 + h_3^3). \end{cases}$$

The above relationship implies that we can sufficiently focus on the former two equations alone in order to determine ω'_2 and ω'_3 .