

Multiple Steady States under the Balanced Budget Rule— a Generalization*

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October 21, 2013

Abstract

When governments levy taxes on labour income on the basis of a balanced budget rule, this rule causes a nonlinear system. Thus, multiple steady states in an economy exist, which can cause multiple movement patterns in an economy. This article deals with the existence of these multiple steady states. Schmitt-Grohé and Uribe (1997) discusses this issue, but does not necessarily show the clear existence of steady states. On a more general assumption of increasing marginal disutility of labour, however, we show that there can be two steady states in the economy, one with superior, the other with inferior economic performance.

JEL classification: C62; E13; E21; E32; H24

Keywords; Multiple steady states; Balanced budget rule; Labour income taxation; Divisible labour

*This article is based on my doctoral dissertation, presented to Kobe University in 2006. I am grateful to Professor Jun Iritani for his persistent guidance. Moreover, I thank Professor Kazuo Mino for his sharp and relevant criticism at an annual academic meeting of the J. E. A. at Kyoto University on June 7, 2009. I should also acknowledge comments from Professor Tamotsu Nakamura. Additionally, I have been advised about the English in this paper by Professor Jeffrey Herrick. I note it with appreciation. Naturally, I accept full responsibility for any errors.

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1 Introduction

In this paper and a subsequent one we demonstrate that there exist two steady states, and correspondingly two different movement patterns, namely, saddle paths and indeterminate ones, in an economy governed by a balanced budget rule, in which labour income taxation is assumed.¹

There is a lot of literature on the issue of indeterminacy. Originally, Black (1974) and Brock (1974) point out that there is a possibility of indeterminacy in a dynamic monetary model with a perfect foresight principle, that is, they assert that no meaningful path can be determined. In this situation, some non-fundamental factors such as expectations, which can be called sunspots or animal spirits, can determine a real economic path. In this sense, indeterminacy is a hot issue. Since then, many economists have located the causes of this indeterminacy externally in production or monopolistic competition. In line with this, we can list Benhabib and Farmer (1994).

Furthermore, Kamihigashi (2002) deals with this issue in terms of externality and nonlinear discounting with respect to utility. Guo and Lansing (1998) and Wen (2001) study the problem in terms of externality and taxation and of externality and depreciation,² respectively. Apparently, these authors mainly seek the causes of indeterminacy in economic conditions, that is, naturally produced economic outcomes.

On the other hand, we would like to discuss indeterminacy from another viewpoint, that is, in relation to fiscal policy, as a product only of artificial determinants. In this context, fiscal policy specifically means labour income taxation and a balanced budget principle. We want to examine whether changes in taxation by governments might cause changes in economic movements.

Schmitt-Grohé and Uribe (1997) examines this issue in an infinite horizon model and on the basis of a government balanced budget rule. And they conclude with an affirmation that indeterminacy can be caused within a certain range of tax rates on labour income in steady states, but not within other tax rates. Their model framework is fundamentally based on Ramsey (1928), but it does not deal with planned economies only, as Ramsey (1928) did, instead extending the scope to equilibria in markets. Therefore, their discussion in relation to indeterminacy or other movement patterns depends

¹In this paper, however, we discuss only the former; the latter will be demonstrated in another article.

²These papers discuss a one-sector model. On the other hand, Benhabib and Farmer (1996) and Guo and Harrison (2001) discuss indeterminacy in a two-sector model.

intensively upon the existence and properties of steady states, which should be derived in their model. However, the derivation is not clear, because they do not present necessary and sufficient conditions for the steady states.

In light of this argument, we deal only with the properties of the steady states based on a new model, which includes Schmitt-Grohé and Uribe (1997) as a special case. Movement patterns including indeterminacy will be dealt with in our subsequent paper.

In Schmitt-Grohé and Uribe's framework, it is necessary to deduce the supply of the labour force.³ The household does not actively determine the supply of labour; instead, the supply of labour provided by the household is supposed to be arbitrary. Thus, the labour supply is determined independently from labour income tax rates, although they are determined endogenously. This is why the utility function is assumed to be linear with regard to labour. And these properties seem unnatural and too elaborate. .

This leads to an important question: what happens when the household feels an increasing marginal disutility with regard to labour? It is indisputable that this postulation will lead to a clear and more natural analytical framework.

Motivated thus, we establish a model in which the household actively determines the supply of labour, under an assumption of an increasing disutility with regard to labour.

In this article, we deal with labour income taxation, in which the tax rates on labour income are flexible and, correspondingly, government expenditure is exogenously constant, under the balanced budget rule. This system causes nonlinear dynamics.⁴

As a result, based on our model, we conclude that there are two steady states, which can be clearly categorized as superior or inferior in terms of economic performance; in addition, that there can be a double steady state; and we demonstrate some conditions for the existence of two steady states, which are established from a viewpoint of labour supply incentives to labour tax rates in *Theorem 1*. More precisely, *Theorem 1* implies that there exist two steady states when government expenditure is less than or equal to the maximum revenue of labour tax. (See the first subsection in

³Neoclassical growth models like Solow (1956) often assume the supply of labour as determined exogenously, but they assume that the supply of labour is given by indivisible entering or leaving employment, and therefore flexible. This assumption implies that the labour supply can extend to infinity, corresponding to demand for labour, if any.

⁴On the other hand, there is an another balanced budget rule in which tax rates remain constant while government expenditure is flexible. And, this system causes linear dynamics (See Guo and Harrison (2004) and Takata (2006)).

section 3.) However, we also obtain a conclusion that our taxation leaves efficiency unchanged; in other words, the same level of capital intensity holds in both steady states, regardless of the corresponding tax rates, which we call a *Neutrality Theorem* (See the second subsection in section 3); and that there are clear conditions under which maximum government revenues are decided, and these conditions are explicitly shown by fixed parameters alone, which indicate fundamentals in an economy.

We organize this article as follows: In the next section, we introduce the framework and approach of the paper, in which the labour supply is deduced explicitly. Of course, it is affected by labour income tax rates. Then we offer a dynamic system. In what follows, in section 3, we demonstrate that there can be two steady states in that system, in which one is superior and the other inferior. Subsequently, section 4 is devoted to a conclusion.

2 A new model

2.1 Assumptions

An economy consists of three kinds of agents, namely households, firms and a government. Households and firms are each assumed to be represented by an agent.

A representative household possesses capital and loans it to a firm in exchange for rental fees, and additionally offers its labour to a firm. The labour-population is constant and households are identical. A household faces a dynamic problem in which it determines an infinite scenario of work and consumption under budget constraints over time.

2.2 Optimisation

First, we assume an instantaneous utility function of a representative household, U_t , as follows:

$$U_t = \log c_t - \frac{H_t^{1-\chi}}{1-\chi}.$$

Here, c_t means consumption and H_t working hours, respectively. A household is assumed to live forever and to maximize lifetime utility. The above utility function has a property of increasing marginal disutility with respect to H_t , contrary to Schmitt-Grohé and Uribe (1997).⁵ Here, χ indicates a

⁵In Schmitt-Grohé and Uribe (1997), an instantaneous utility is expressed as $U_t = \log c_t - AH_t$, which apparently shows a constant marginal disutility of labour, corresponding to $\chi = 0$ in our utility function.

negative constant, so that $-1/\chi$ shows the elasticity of the labour supply with respect to wages, which will be proved later.

For simplicity, we postulate that the duration of capital is infinite, and as a result we omit depreciation.

In this situation, a household considers labour tax rates, τ_t , real wage rates, w_t , and real rental rates of capital, u_t , as given, and solves the following dynamic problem:

$$\max_{c_t, H_t} \int_0^\infty e^{-\rho t} \left[\log c_t - \frac{H_t^{1-\chi}}{1-\chi} \right] dt,$$

subject to

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - c_t, \quad t \in \mathfrak{R}_+. \quad (1)$$

Here, \mathfrak{R}_+ is the set of positive reals. Additionally, ρ indicates a subjective discount rate of utility.

Second, the government observes a balanced budget rule. Since the government can observe a tax base, that is, $w_t H_t$, through the labour market, it determines a tax rate, τ_t , in order to make the labour taxes levied equal to constant government spending, G , which a balanced budget rule requires. This can be formulated as

$$G = \tau_t w_t H_t. \quad (2)$$

Third, we discuss firms.

We introduce a production function F , a Cobb-Douglas type, as

$$F(K_t, H_t) = K_t^\alpha H_t^{1-\alpha},$$

where α is a constant satisfying $0 < \alpha < 1$. We assume that the markets for labour, capital, and output are fully competitive, and that a firm aims to maximise its profit. As a result, the following relationships hold:

$$w_t = \frac{\partial F}{\partial H} = F_H(K_t, H_t) = (1 - \alpha) \left(\frac{K_t}{H_t} \right)^\alpha,$$

and

$$u_t = \frac{\partial F}{\partial K} = F_K(K_t, H_t) = \alpha \left(\frac{K_t}{H_t} \right)^{\alpha-1}.$$

Here, output price is assumed to be 1.

At this stage, we define a Hamiltonian as

$$R = e^{-\rho t} \left[\log c_t - \frac{H_t^{1-\chi}}{1-\chi} \right] + \mu_t [u_t K_t + (1 - \tau_t) w_t H_t - c_t].$$

We obtain the following necessary conditions for optimisation:

$$\frac{\partial R}{\partial c_t} = e^{-\rho t} \frac{1}{c_t} - \mu_t = 0,$$

which can be transformed as

$$\mu_t = \frac{e^{-\rho t}}{c_t}. \quad (3)$$

On the other hand,

$$\frac{\partial R}{\partial H_t} = e^{-\rho t} \left[-H_t^{-\chi} \right] + \mu_t (1 - \tau_t) w_t = 0$$

holds. Contrary to Schmitt-Grohé and Uribe (1997), our present maximisation determines the supply of labour. In short, the above equality, together with (3), enables us to know:

$$H_t = \left[\frac{c_t}{(1 - \tau_t) w_t} \right]^{\frac{1}{\chi}}. \quad (4)$$

Based on (4), we can easily confirm that the elasticity of this labour supply with respect to wages, e , equals $-1/\chi$.

Furthermore, the following holds concerning the adjoint variable μ_t :

$$\frac{d\mu_t}{dt} = -\frac{\partial R}{\partial K_t} = -\mu_t u_t,$$

which yields

$$\frac{\dot{\mu}_t}{\mu_t} = -u_t.$$

Based on the above relationship and (3), we obtain an Euler equation:

$$\frac{\dot{c}_t}{c_t} = u_t - \rho.$$

Here, we should notice that the Euler equation does not include tax rate τ_t .

Moreover, for the household to project its budget constraints over time, the transversality condition needs to be satisfied,⁶ which can be expressed as $\lim_{t \rightarrow \infty} \mu_t k_t = 0$. This can be further transformed as

$$\lim_{t \rightarrow \infty} \frac{k_t}{c_t} e^{-\rho t} = 0.$$

Here, we define the capital intensity of labour as $k_t = K_t/H_t$.

Of course, the transversality condition means that the present value of the marginal utility of consumption at infinity in terms of capital stock, which can be transformed into consumption through perfect substitution, must be null.⁷

In addition, since R is concave with respect to c_t , H_t , μ_t and K_t , the conditions sufficient for the objective function are satisfied. In short, solutions satisfying necessary conditions are solutions for optimisation.

2.3 Dynamics

Now, we need to derive the dynamic system of our economy.

First, in light of $k_t = K_t/H_t$ we obtain the following, from (1):

$$\dot{k}_t = u_t \frac{K_t}{H_t} + (1 - \tau_t)w_t - \frac{c_t}{H_t} - k_t \frac{\dot{H}_t}{H_t}. \quad (5)$$

Second, based on (4), we can express the rate of change in the labour supply as

$$\frac{\dot{H}_t}{H_t} = \frac{1}{\chi} \frac{\frac{\dot{c}_t}{c_t} \frac{1 - \tau_t}{\tau_t} + \frac{\dot{\tau}_t}{\tau_t} - \frac{1 - \tau_t}{\tau_t} \frac{\dot{w}_t}{w_t}}{\tau_t}. \quad (6)$$

Here, let us consider government behavior, which is clearly described by (2), and which can be transformed as

$$\frac{\dot{\tau}_t}{\tau_t} = -\frac{\dot{w}_t}{w_t} - \frac{\dot{H}_t}{H_t}. \quad (7)$$

Substituting (7) into (6) and rearranging the result, we eventually obtain the rate of change in the labour supply as

$$\frac{\dot{H}_t}{H_t} = \frac{1}{(1 - \chi) \tau_t + \chi} \left(\frac{\dot{c}_t}{c_t} (1 - \tau_t) - \frac{\dot{w}_t}{w_t} \right). \quad (8)$$

⁶We assume that a household eventually dies at infinity, and that capital stock can be transformed into consumption with perfect substitution.

⁷See Blanchard and Fischer (1989).

Additionally, we postulate naturally that $(1 - \chi) \tau_t + \chi \neq 0$.

Now, let us begin to calculate \dot{k}_t , which is described in (5). Then we can obtain a dynamic equation with regard to k_t as

$$\begin{aligned} \frac{\dot{k}_t}{k_t} = & \frac{(\alpha + (1 - \alpha)(1 - \tau_t))((1 - \chi) \tau_t + \chi) - \alpha (1 - \tau_t)}{(1 - \chi) \tau_t + \chi - \alpha} k_t^{-(1-\alpha)} \\ & - \frac{c_t^{1-\frac{1}{\chi}} (1 - \tau_t)^{\frac{1}{\chi}} (1 - \alpha)^{\frac{1}{\chi}} ((1 - \chi) \tau_t + \chi)}{(1 - \chi) \tau_t + \chi - \alpha} k_t^{\frac{\alpha}{\chi}-1} \\ & + \frac{\rho (1 - \tau_t)}{(1 - \chi) \tau_t + \chi - \alpha}. \end{aligned} \quad (9)$$

Additionally, we postulate $(1 - \chi) \tau_t + \chi - \alpha \neq 0$.

In the above procedures, we utilize the following:

$$\begin{cases} c_t \left[\frac{c_t}{(1 - \tau_t) w_t} \right]^{-\frac{1}{\chi}} = c_t^{1-\frac{1}{\chi}} [(1 - \tau_t) w_t]^{\frac{1}{\chi}}, \\ \frac{\dot{w}_t}{w_t} = \alpha \frac{\dot{k}_t}{k_t}, \\ w_t = (1 - \alpha) k_t^\alpha, \\ \frac{\dot{c}_t}{c_t} = u_t - \rho = \alpha k_t^{-(1-\alpha)} - \rho. \end{cases} \quad (10)$$

Next, we explore a dynamic equation with regard to τ_t , which plays a critical role in our system, as will be revealed later.

Based on (7) and (8), we obtain the following:

$$\begin{aligned} \frac{\dot{\tau}_t}{\tau_t} = & - \frac{\dot{w}_t}{w_t} - \frac{1}{(1 - \chi) \tau_t + \chi} \left(\frac{\dot{c}_t}{c_t} (1 - \tau_t) - \frac{\dot{w}_t}{w_t} \right) \\ = & \frac{1 - \tau_t}{(1 - \chi) \tau_t + \chi} \left[\alpha (1 - \chi) \frac{\dot{k}_t}{k_t} - \frac{\dot{c}_t}{c_t} \right]. \end{aligned} \quad (11)$$

Based on (9) and (10), (11) can be transformed into (12):

$$\begin{aligned} \frac{\dot{\tau}_t}{\tau_t} = & \frac{1 - \tau_t}{(1 - \chi) \tau_t + \chi} \left[\alpha \frac{(\chi + (1 - \chi) \tau_t)((\alpha - \chi) - (1 - \chi)(1 - \alpha) \tau_t)}{(1 - \chi) \tau_t + \chi - \alpha} k_t^{\alpha-1} \right. \\ & - \frac{\alpha (1 - \chi) c_t^{1-\frac{1}{\chi}} (1 - \tau_t)^{\frac{1}{\chi}} (1 - \alpha)^{\frac{1}{\chi}} ((1 - \chi) \tau_t + \chi)}{(1 - \chi) \tau_t + \chi - \alpha} k_t^{\frac{\alpha}{\chi}-1} \\ & \left. + \frac{\rho (1 - \alpha) ((1 - \chi) \tau_t + \chi)}{(1 - \chi) \tau_t + \chi - \alpha} \right]. \end{aligned} \quad (12)$$

As a consequence, we can establish our dynamic system consisting of (9), the Euler equation and (12).

At this stage, we introduce new variables:

$$\begin{cases} \log k_t = \lambda_t, \\ \log c_t = \delta_t, \\ \log \tau_t = \eta_t. \end{cases}$$

In this situation, the system consisting of (9), the Euler equation and (12) can be expressed as follows:

$$\dot{\delta}_t = \alpha e^{-(1-\alpha)\lambda_t} - \rho. \quad (13)$$

From this we can extrapolate the following:

$$\begin{aligned} \dot{\lambda}_t = & \frac{(\alpha + (1-\alpha)(1-e^{\eta_t}))((1-\chi)e^{\eta_t} + \chi) - \alpha(1-e^{\eta_t})}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\alpha-1)\lambda_t} \\ & - \frac{e^{\frac{\chi-1}{\chi}\delta_t}(1-e^{\eta_t})^{\frac{1}{\chi}}(1-\alpha)^{\frac{1}{\chi}}((1-\chi)e^{\eta_t} + \chi)}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\frac{\alpha}{\chi}-1)\lambda_t} \\ & + \frac{\rho(1-e^{\eta_t})}{(1-\chi)e^{\eta_t} + \chi - \alpha}, \end{aligned} \quad (14)$$

and

$$\begin{aligned} \dot{\eta}_t = & \frac{1-e^{\eta_t}}{(1-\chi)e^{\eta_t} + \chi} \left[\frac{\alpha(\chi + (1-\chi)e^{\eta_t})((\alpha-\chi) - (1-\chi)(1-\alpha)e^{\eta_t})}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\alpha-1)\lambda_t} \right. \\ & - \frac{\alpha(1-\alpha)^{\frac{1}{\chi}}(1-\chi)e^{\frac{\chi-1}{\chi}\delta_t}(1-e^{\eta_t})^{\frac{1}{\chi}}((1-\chi)e^{\eta_t} + \chi)}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\frac{\alpha}{\chi}-1)\lambda_t} \\ & \left. + \frac{\rho(1-\alpha)((1-\chi)e^{\eta_t} + \chi)}{(1-\chi)e^{\eta_t} + \chi - \alpha} \right]. \end{aligned} \quad (15)$$

This system, (13) through (15), consists of three variables, λ_t , δ_t and η_t and the three corresponding differential equations. Notice that the system does not include government spending G .

3 Existence of multiple equilibria

3.1 Two equilibria

Assume that there is a steady state in our system, denoted by λ^* , δ^* and η^* .

First, by making $\dot{\delta}_t = 0$ in (13), we obtain

$$k^* = e^{\lambda^*} = \left(\frac{\alpha}{\rho} \right)^{\frac{1}{1-\alpha}}. \quad (16)$$

Second, in a similar fashion, as $\dot{\delta}_t = 0$, by making $\dot{\lambda}_t = 0$ in (14) and $\dot{\eta}_t = 0$ in (15), we obtain the following:

When $\dot{\lambda}_t = 0$,

$$\begin{aligned} & [(\alpha + (1 - \alpha)x)((\chi - 1)x + 1) - \alpha x] \left(\frac{\rho}{\alpha} \right) \\ & - x^{\frac{1}{\chi}} (1 - \alpha)^{\frac{1}{\chi}} ((\chi - 1)x + 1) q e^{\frac{\alpha - \chi}{\chi} \lambda^*} + \rho x = 0, \end{aligned} \quad (17)$$

and when $\dot{\eta}_t = 0$,

$$\begin{aligned} & \alpha (\chi + (1 - \chi)(1 - x))((\alpha - \chi) - (1 - \chi)(1 - \alpha)(1 - x)) \left(\frac{\rho}{\alpha} \right) \\ & - \alpha (1 - \alpha)^{\frac{1}{\chi}} (1 - \chi) q x^{\frac{1}{\chi}} ((\chi - 1)x + 1) e^{\frac{\alpha - \chi}{\chi} \lambda^*} \\ & + \rho (1 - \alpha)((\chi - 1)x + 1) = 0. \end{aligned} \quad (18)$$

Additionally, to have a clear perspective, we transform variables as

$$\begin{cases} 1 - e^{\eta^*} = x, \\ e^{\frac{\chi - 1}{\chi} \delta^*} = q. \end{cases} \quad (19)$$

Here, because $0 < e^{\eta^*} = 1 - x = \tau^* < 1$, $0 < x < 1$. In this situation, we consider x and q as variables.

We consider the above system, (17) and (18), as a simultaneous equation system concerning x and q .

However, we can easily confirm that $(17) \times \alpha(1 - \chi) = (18)$. This implies that (17) and (18) are not independent of each other. Therefore, we seek another relationship including x and q .

We need to focus on the balanced budget in a steady state, which is described as

$$G = \tau^* w^* H^* = \tau^* w^* \left[\frac{c^*}{(1 - \tau^*) w^*} \right]^{\frac{1}{\chi}}. \quad (20)$$

Notice that (18) is originally deduced from the balanced budget rule, and therefore the dynamic equation concerning τ is not independent (See (11)).

Rearranging (20), we obtain the following:

$$q = \left[\frac{G^\chi}{(w^*)^{\chi-1}} \frac{x}{(1-x)^\chi} \right]^{\frac{\chi-1}{\chi}}. \quad (21)$$

Substituting (21) into (17), we obtain an equation only in terms of x as

$$\begin{aligned} & [(\alpha+(1-\alpha)x)((\chi-1)x+1) - \alpha x] \left(\frac{\rho}{\alpha}\right) + \rho x \\ & - (1-\alpha)^{\frac{1}{\chi}} e^{\frac{\alpha-\chi}{\chi}\lambda^*} ((\chi-1)x+1) \\ & \times x^{\frac{1}{\chi}} \left[\frac{G^\chi}{(w^*)^{\chi-1}} \right]^{\frac{\chi-1}{\chi}} \left[\frac{x}{(1-x)^\chi} \right]^{\frac{\chi-1}{\chi}} = 0, \end{aligned} \quad (22)$$

which can further be transformed as

$$\begin{aligned} & [(\alpha+(1-\alpha)x)((\chi-1)x+1) - \alpha x] \left(\frac{\rho}{\alpha}\right) + \rho x \\ & - G^{\chi-1} (1-\alpha)^{2-\chi} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(2-\chi)-1}{1-\alpha}} ((\chi-1)x+1)(1-x)^{1-\chi} x = 0. \end{aligned} \quad (23)$$

When we denote the left side in the above equation as $h(x)$, that can be expressed as

$$\begin{cases} h(x) = (\chi-1) \left(x - \frac{1}{1-\chi}\right) (1-\alpha)\varphi(x), \\ \varphi(x) = x [1 - B(1-x)^{1-\chi}] + \frac{\alpha}{1-\alpha}, \\ B = (1-\alpha)^{1-\chi} G^{\chi-1} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(1-\chi)}{1-\alpha}} = \left(\frac{w^*}{G}\right)^{1-\chi} > 0. \end{cases} \quad (24)$$

The solution in the above equation, $h(x) = 0$, shows the steady state which we are investigating. Of course, $0 < x < 1$ must be satisfied.

From what (24) implies, since x which satisfies $h(x) = 0$ is a possible solution, $x = 1/(1-\chi)$ seems to be a solution on appearance. However, if $x = 1/(1-\chi)$ holds, $(1-\chi)\tau^* + \chi = 0$ is satisfied, which we have rejected (See (8)).

At this stage, it is obvious that the steady state is a solution satisfying $\varphi(x) = 0$.

Therefore, we need to examine how $\varphi(x)$ behaves in the range of $0 < x < 1$. We know that the curve has a minimum point at $0 < x^* < 1$. So, if the curve at the minimum point is negative, our system can have two steady

states.⁸

Based on Appendix 1, we offer an example in Figure 1.

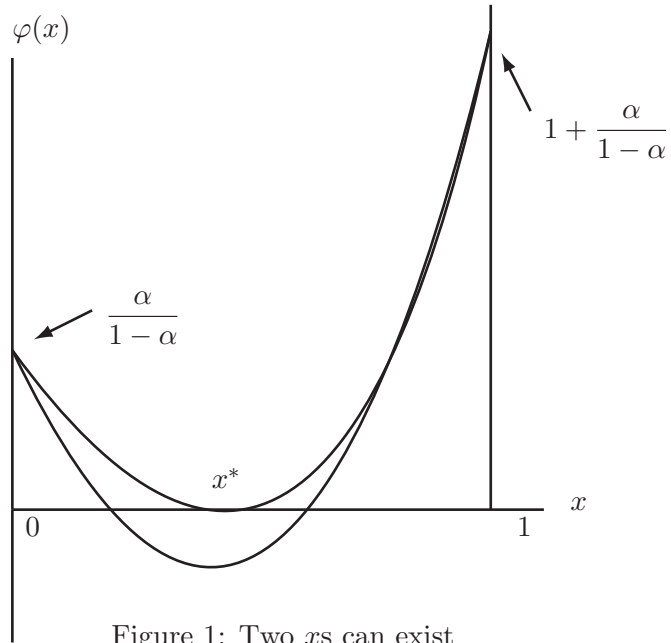


Figure 1: Two x s can exist

As we have discussed so far, assuming an increasing marginal disutility with regard to labour, we can offer some conditions for multiple steady states as a theorem:

Let x^* be a solution to the following equation, as is demonstrated in Appendix 1:

$$1 - (2 - \chi)x^* = \frac{1}{B(1 - x^*)^{-\chi}}. \quad (25)$$

In this situation, we can show the existence of multiple steady states.

Theorem 1 (The existence of two steady states) *There exist two steady states if the following conditions are satisfied:*

⁸For details, see Appendix 1. Additionally, there is a possibility that the system has a double steady state. We deal with this issue in the second subsection of this section.

$$\begin{cases} (a) & B > 1, \\ (b) & x^*[1 - B(1 - x^*)^{1-\chi}] + \frac{\alpha}{1-\alpha} \leq 0, \\ (c) & K^\alpha H^{1-\alpha} = c + G. \end{cases}$$

Additionally, condition (c) reflects Walras' law and shows the values in the steady states.

When a unique G is given, in the case of inequality in (b), there can correspondingly be two steady states, while in the case of equality in (b), a double root in relation to x^* appears. This implies that two similar steady states can coexist. In the second subsection in this section, we show that the system satisfying *Theorem 1* is not empty.

Theorem 1 is established based on a particular x^* satisfying (25). However, we can demonstrate the same relationship as *Theorem 1* for an arbitrary situation in which $0 < x < 1$.

Corollary 1 (The existence of two steady states) *There exist two steady states if the following conditions are satisfied:*

$$\begin{cases} (d) & B = \frac{x + \frac{\alpha}{1-\alpha}}{x(1-x)^{1-\chi}}, \\ (e) & K^\alpha H^{1-\alpha} = c + G. \end{cases}$$

Since the right hand side of (d) in *Corollary 1* is larger than unity for any value of $0 < x < 1$, the condition $B > 1$ in (a) in *Theorem 1* is necessarily satisfied in *Corollary 1*. (A proof is given in Appendix 2) It is not clear that Schmitt-Grohé and Uribe demonstrate or admit the equality in (b) in *Theorem 1*, or the existence of a double steady state. However, the discussions both in (d) and in Appendix 2 can be applicable to the case of $\chi = 0$, as in Schmitt-Grohé and Uribe (1997). This is why our model is a generalization which includes Schmitt-Grohé and Uribe (1997) as a special case.

In this situation, we need to elucidate (d) in more detail, from an economic viewpoint. Transforming (d), we obtain

$$G = w^*(1-x) \left(\frac{x}{x + \frac{\alpha}{1-\alpha}} \right)^{\frac{1}{1-\chi}}. \quad (26)$$

The right side in (26) shows labour tax revenue in terms of x alone⁹ and independent of G . Now, let us focus on the shape of this labour tax revenue

⁹Note that w^* is constant.

function $\zeta(\tau^*) = w^* \tau^* h(\tau^*)$, where $h(\tau^*)$ is deduced as

$$h(\tau^*) = \left(\frac{1 - \tau^*}{1 - \tau^* + \frac{\alpha}{1-\alpha}} \right)^{\frac{1}{1-\chi}},$$

which indicates the labour supply in terms of τ^* alone. Note that $x = 1 - \tau^*$.

Based on the definition of $\zeta(\tau^*)$, we obtain the following:

$$\frac{d\zeta}{d\tau^*} = w^* h(\tau^*) \left(1 + \frac{\tau^*}{h(\tau^*)} \frac{dh(\tau^*)}{d\tau^*} \right),$$

where we define $\epsilon(\tau^*) = \frac{\tau^*}{h(\tau^*)} \frac{dh(\tau^*)}{d\tau^*}$. $\epsilon(\tau^*)$ means the elasticity of labour supply in relation to labour income tax rates and has the following properties: $\epsilon'(\tau^*) = \frac{-\frac{\alpha}{1-\alpha}\tau^*}{(1-\tau^*)(1-\chi)(1-\tau^* + \frac{\alpha}{1-\alpha})} < 0$; $\epsilon(0) = 0$; $\epsilon(1) \rightarrow -\infty$. This implies that as τ^* increases, the labour supply decreases drastically. Therefore, when $0 < \tau^* \leq \tau^{**}$, $\epsilon + 1 \geq 0$, while in ranges of $\tau^{**} < \tau^* < 1$, $\epsilon + 1 < 0$. This indicates that when $0 < \tau^* \leq \tau^{**}$, $\frac{d\zeta}{d\tau^*} \geq 0$, and when $\tau^{**} < \tau^* < 1$, $\frac{d\zeta}{d\tau^*} < 0$.

Additionally, τ^{**} shows the labour income tax rate corresponding to maximum tax revenue. Furthermore, $\zeta(\tau^*)$ has the following properties: $\zeta(0) = 0$; $\zeta(1) = 0$ and $\zeta > 0$, for $0 < \tau^* < 1$. And ζ is a continuous function with respect to τ^* .

From what we have described so far, it is obvious that the shape of ζ is like that of a bell curve.¹⁰ This bell curve is independent of G . Therefore, we can say that there are two τ^* s when G is given. In other words, there are two alternatives in relation to τ^* which correspond to the government spending financed by this tax revenue. Theorem 1 shows that there is a bell curve in our system, which causes nonlinear dynamics. In this context, G must naturally be less than or equal to the maximum tax revenue, which is determined by fundamental parameters alone. (See the following subsection) This is what *Corollary 1* shows. In other words, G is restricted by the fundamentals in the system.

¹⁰Schmitt-Grohé and Uribe deduce this kind of bell curve avoiding H itself, instead considering K , which is a function with τ^* as an argument. (See Schmitt-Grohé and Uribe (1997), page 981).

3.2 A small τ^* is preferable

In this situation, the two x s imply that the government can choose two τ^* s, that is, a lower τ^* and a higher τ^* . The former leads to superior economic performance, the latter to an inferior one, because in the case of a lower τ^* a household enjoys higher utility, while in the case of a higher τ^* it endures a lower one.

Clearly, when the government imposes a higher tax rate on a household, the household's disposable income is significantly reduced, compared with a lower tax rate. And this depresses our economy to a large extent through income effects.

We can confirm this as follows:

First, we can express H^* and c^* in terms of a function with an argument about x alone, as below.

$$H(x) = \left(\frac{x}{x+A} \right)^{\frac{1}{1-\chi}} = \left[\frac{c^*}{x w^*} \right]^{\frac{1}{\chi}}$$

and based on the above relationship, c^* can be deduced as

$$c(x) = \left(\frac{x}{x+A} \right)^{\frac{\chi}{1-\chi}} w^* x.$$

Then, we can show U as a function with an argument about x alone,

$$U(x) = \frac{\chi}{1-\chi} \text{Log} \left(\frac{x}{x+A} \right) + \text{Log}(w^*) + \text{Log}(x) - \frac{1}{1-\chi} \left(\frac{x}{x+A} \right).$$

In this situation, our aim is to deduce $\partial U / \partial x > 0$. We easily verify this as

$$\frac{\partial U}{\partial x} = \frac{A+x+\frac{A^2}{x-x\chi}}{(A+x)^2} = \frac{(1-\chi)x^2 + (1-\chi)Ax + A^2}{x(A+x)^2(1-\chi)} > 0, \text{ for } 0 < x < 1.$$

This shows that the utility of a household decreases when τ^* increases, and vice versa. Because we consider which utility is larger, corresponding to the two τ^* s, it is obvious that a lower τ^* leads to larger utility, a higher τ^* to a smaller one.

Second, we proceed to verify that a comparatively higher τ^* reduces H^* , c^* , K^* and Y^* more intensively. As for the relationship between H^* and τ^* , we have a clear conclusion. Thus, we analyse the others. The variables are expressed for x alone in the following way: $c^* = \left(\frac{x}{x+A} \right)^{\frac{\chi}{1-\chi}} w^* x$; $K^* = k^* H^* = \left(\frac{\alpha}{\rho} \right)^{\frac{1}{1-\alpha}} \left(\frac{x}{x+A} \right)^{\frac{1}{1-\chi}}$; $Y^* = \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha}{1-\alpha}} \left(\frac{x}{x+A} \right)^{\frac{1}{1-\chi}}$.

By simple analyses, we can obtain the following: $\frac{\partial c^*}{\partial x} = -\frac{w^*\left(\frac{x}{A+x}\right)^{\frac{\chi}{1-\chi}}(A+x(1-\chi))}{(A+x)(-1+\chi)} > 0$.

Additionally, since $d\left(\frac{x}{x+A}\right)^{\frac{1}{1-\chi}}/dx = \frac{A}{(A+x)^2} > 0$, it is obvious that K^* and Y^* increase when x does.

Furthermore, $c^* + G = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \left(\frac{x}{x+A}\right)^{\frac{1}{1-\chi}} = Y^*$ holds. Since this shows that the two τ^* s satisfy Walras' law, it is obvious that there are two steady states in our economy. In addition, notice that both hold under the same capital-labour intensity, k^* , and under different income tax rates, τ^* s.

We call this a *neutrality theorem*.

Theorem 2 (Neutrality theorem) *Labor income taxation does not affect economic efficiency in steady states.*

The reason is as follows:

$k^* = (\alpha/\rho)^{\frac{1}{1-\alpha}}$ is proved. (See (16)) This evidently shows that k^* is not affected by τ .

This *Theorem* shows w^* and u^* are not also affected by τ^* s.

3.3 Maximum G

At this stage, let us focus on the relationship between G and our two x s. We can easily know from the above analyses that if G increases, $\varphi(x)$ tends to go upwards, which is caused by a decrement of B . And this leads to a double root or no root with regard to x . This implies an upper limitation with regard to G in our context. Of course, maximum G corresponds to maximum labour income tax revenue.

We can find the maximum G as follows:

First, since $\varphi'(x^*) = 0$ must be satisfied, we obtain

$$B = \frac{(1-x^*)^\chi}{1-(2-\chi)x^*}. \quad (27)$$

Second, under the condition of the above B , $\varphi(x^*) = 0$ must be also satisfied. Therefore, the following relationship holds:

$$x^* \left[1 - \frac{(1-x^*)^\chi(1-x^*)^{1-\chi}}{1-(2-\chi)x^*} \right] + \frac{\alpha}{1-\alpha} = 0,$$

which can be transformed as

$$(\chi-1)(x^*)^2 - \frac{\alpha(2-\chi)}{1-\alpha}x^* + \frac{\alpha}{1-\alpha} = 0.$$

Solving the above quadratic equation, we eventually obtain the solution required as

$$x^* = \frac{\alpha(2 - \chi) - \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1 - \alpha)(\chi - 1)} > 0. \quad (28)$$

Substituting (28) into (27), B is determined by parameters such as α and χ alone, and this shows the maximum G . Thus,

$$G = (1 - \alpha) \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha}{1 - \alpha}} \left[\frac{1 - (2 - \chi)x^*}{(1 - x^*)^\chi} \right]^{\frac{1}{1 - \chi}}$$

is derived. Moreover, in this context, we can demonstrate that the values in this equilibrium of c , τ and H are explicitly determined by fundamental parameters alone, such as α , ρ and χ as G .

We offer only the results of this determination as follows:

$$\begin{cases} \tau^* = \frac{\chi(2 - \alpha) - 2 + \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1 - \alpha)(\chi - 1)}, \\ G = (1 - \alpha) \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha}{1 - \alpha}} \left[\frac{(\tau^*)^\chi}{1 - (2 - \chi)(1 - \tau^*)} \right]^{-\frac{1}{1 - \chi}}, \\ c^* = (1 - \alpha) \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha}{1 - \alpha}} (1 - \tau^*) \left[\frac{\tau^*}{1 - (2 - \chi)(1 - \tau^*)} \right]^{-\frac{\chi}{1 - \chi}}, \\ H^* = \left[\frac{\tau^*}{1 - (2 - \chi)(1 - \tau^*)} \right]^{-\frac{1}{1 - \chi}}. \end{cases}$$

From what we have analyzed so far, it becomes obvious that w^* , c^* , G and H^* can be expressed by parameters alone. Of course, so can k^* . Thus, we can easily describe K^* , u^* and Y^* by parameters alone as well.

4 Conclusion

We have explored the possibility of multiple steady states under a balanced budget rule, which Schmitt-Grohé and Uribe (1997) leaves ambiguous. We have unambiguously demonstrated a crucial issue, that there are two steady states. These phenomena characteristically occur under economic conditions in which fundamental factors such as production technologies, utility functions and attitudes with regard to the labour supply remain unchanged.

We postulate a utility function which has a property of increasing marginal disutility with regard to labour supply, while Schmitt-Grohé and Uribe

(1997) posits constant marginal disutility with regard to labour supply. And our model generalizes Schmitt-Grohé and Uribe (1997). Due to our postulation, we can specifically deduce a labour supply function, and it becomes obvious that the shape of the tax revenue function is like a bell curve with an independent argument regarding tax rates at steady states. Moreover, this shows that our system is nonlinear. With the help of global analysis, we demonstrate that this causes the existence of two steady states. And this offers the government two alternatives regarding tax rate policy. Higher or lower rates can be imposed in order to levy similar tax revenues. But these two options result in different economic performance, in terms of production scales, consumption, and especially in terms of utility levels. Higher rates produce inferior results compared to lower rates.

However, the intensity of capital to labour is the same in both steady states. This shows that production efficiency is the same in both equilibria, and that, as a result, real wage rates and real rental rates are also the same as in the two equilibria. This is why labour income taxation does not affect the relative prices of consumption, either in the present or the future. But this taxation affects savings through income effects.

Moreover, we demonstrate that in our economy, there is a maximum tax revenue which is determined by fundamental structures alone, such as indications of production efficiency, α , plus χ , which shows attitudes regarding the labour supply when wages change, and ρ , which shows a subjective discount rate of utility. And we have demonstrated that at this maximum tax revenue, a double steady state appears.

In light of the two steady states noted, we can easily anticipate that multiple movement patterns can coexist in an economy. Concerning this interesting issue, we obtain conclusions different from Schmitt-Grohé and Uribe (1997). However, we deal with them in a subsequent paper.

Appendix 1

Let us consider a function $\varphi(x)$, which was defined as

$$\varphi(x) = x[1 - B(1 - x)^{1-\chi}] + \frac{\alpha}{1 - \alpha}.$$

First, we focus on the contents in the bracket above, here defined as $f(x) = 1 - B(1 - x)^{1-\chi}$.

We can easily obtain the following information with regard to $f(x)$: $f'(x) = B(1 - x)^{-\chi} > 0$, $f(0) = 1 - B \geq 0$ and $f(1) = 1$.

Based on the above information, we can say that if $B \leq 1$, $f \geq 0$, and that therefore, since $\varphi(x) > 0$, there is no steady state. So, we assume $B > 1$.¹¹

Differentiating $\varphi(x)$ with regard to x , we obtain the following:

$$\begin{aligned}\varphi'(x) &= B(1-x)^{-\chi}[(2-\chi)x-1] + 1 \\ &= B(1-x)^{-\chi}[(2-\chi)x-1 + \frac{1}{B}(1-x)^\chi].\end{aligned}$$

Focusing on the contents in the second bracket in the above relationships, we investigate how $\varphi'(x)$ behaves.

We define $\eta_1(x) = \left(\frac{1}{B}\right)(1-x)^\chi$, which has properties such as $\eta_1'(x) = -\left(\frac{1}{B}\right)\chi(1-x)^{\chi-1} > 0$, $\eta_1(0) = \frac{1}{B} > 0$ and $\eta_1(1) = \infty$.

On the other hand, we define $\eta_2(x) = 1 - (2-\chi)x$, which is monotonously decreasing with regard to x . Additionally, $\eta_2(0) = 1$ and $\eta_2(1) = \chi - 1 < 0$.

Depicting the above relationships, we obtain the following:

¹¹Here, we should consider this relationship from an economic viewpoint. B was defined as follows:

$B = \left(\frac{w^*}{G}\right)^{1-\chi}$. Then, since $\chi < 0$, the following relationship, that is, $w^* > G$, must hold. In the context of this article, w^* indicates a real wage rate at a steady state, which shows output in exchange for a unit of working time. On the other hand, G indicates total wage income taxes levied when working hours equal H^* and the tax rates are τ^* . $B > 1$ shows that the former is larger than the latter.

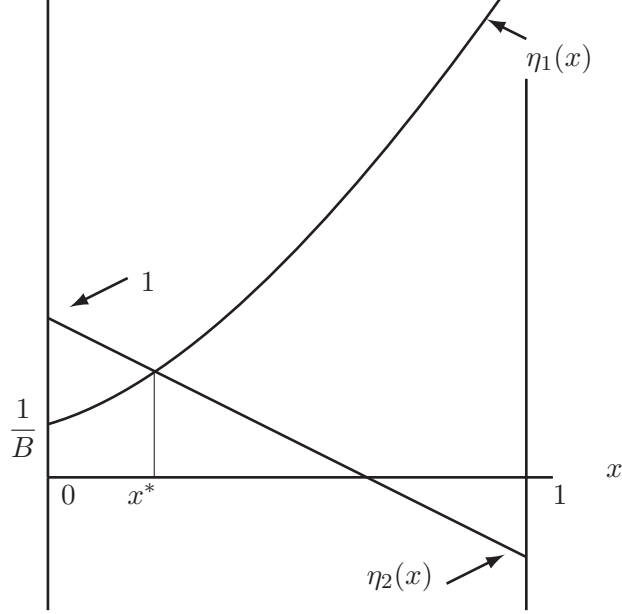


Figure 2: x^* exists uniquely

When G is given, x^* is naturally determined as $1 - (2 - \chi)x^* = \frac{1}{B(1-x^*)^{-\chi}}$.

From the above figure, it becomes obvious that $\varphi(x)$ monotonously decreases when $0 < x < x^*$, and increases when $x^* < x < 1$.

Additionally, $\varphi(x)$ has properties such as $\varphi(0) = \alpha/(1 - \alpha) > 0$ and $\varphi(1) = 1 + \alpha/(1 - \alpha) > 0$.

Appendix 2

From (d) in *Corollary 1* the following can hold;

$$B = \frac{x + \frac{\alpha}{1-\alpha}}{x(1-x)^{1-\chi}} = \frac{x + A}{x(1-x)^\Omega}.$$

Here, we denote $\Omega = 1 - \chi$, $A = \frac{\alpha}{1-\alpha}$, which respectively lead to $\Omega > 1$ and $0 < A < \infty$. Then we express the above mathematical formula as $\xi(x)$. It is obvious that $\xi(0) \rightarrow \infty$ and $\xi(1) \rightarrow \infty$. Differentiating $\xi(x)$ with regard to x , we obtain

$$\xi'(x) = -\frac{(1-x)^{-1-\Omega}(A - Ax - x(A+x)\Omega)}{x^2}.$$

Therefore, we obtain x satisfying $\xi'(x) = 0$ as $\frac{-A(1+\Omega) + \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{2\Omega}$. Then, substituting the above x into $\xi(x)$, we can obtain the minimum point in $\xi(x)$ as

$$\Psi(A, \Omega) = 2^{-1+\Omega} \left(\frac{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{\Omega} \right)^{-\Omega} \times \left(2 + A + A\Omega + \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2} \right).$$

Our aim is to investigate whether $\Psi(A, \Omega) > 1$ holds under $A > 0$ and $\Omega > 1$. We need to explore necessary conditions by differentiating Ψ concerning A and Ω . The following holds:

$$\frac{\partial \Psi}{\partial A} = \frac{1}{\sqrt{A}} 2^{-1+\Omega} \left(\sqrt{A}(1+\Omega) + \sqrt{4\Omega + A(1+\Omega)^2} \right) \times \left(\frac{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{\Omega} \right)^{-\Omega}$$

and

$$\frac{\partial \Psi}{\partial \Omega} = 2^{-1+\Omega} \left(\frac{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{\Omega} \right)^{-\Omega} \times \left(2 + A + A\Omega + \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2} \right) \times \text{Log} \left[\frac{2\Omega}{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}} \right].$$

At this stage, as a preparation for later analyses, first, we can confirm that $(2\Omega + A(1+\Omega))^2 - \left(\sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2} \right)^2 = 4(1+A)\Omega^2 > 0$. This eventually implies that $2\Omega + A(1+\Omega) > \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}$. And then $\frac{\partial \Psi}{\partial A} > 0$ holds, regardless of Ω .

Second, let us proceed to analyse $\frac{\partial \Psi}{\partial \Omega}$. Apparently, the signs of $\frac{\partial \Psi}{\partial \Omega}$ depend on the last item. In general, if Γ is larger than 1, $\frac{\partial \Psi}{\partial \Omega} > 0$. Additionally,

$$\Gamma = \frac{2\Omega}{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}.$$

Here, considering a reciprocal of Γ , the following holds:

$$\frac{1}{\Gamma} = 1 + \frac{A(1 + \Omega) - \sqrt{A}\sqrt{4\Omega + A(1 + \Omega)^2}}{2\Omega}.$$

At this stage, we can verify from $\xi'(x) = 0$

$$0 < \frac{-A(1 + \Omega) + \sqrt{A}\sqrt{4\Omega + A(1 + \Omega)^2}}{2\Omega} < 1.$$

This eventually refers to $\Gamma > 1$. Based on the above discussions, we obtain the conclusion $\frac{\partial \Psi}{\partial \Omega} > 0$.

Let us consider a two-dimensional plane, which has a horizontal line depicting A , and a vertical one depicting Ψ . In this situation, Ψ increases as A does, while Ψ shifts upwards as Ω increases. On the other hand, $\Psi(0, \Omega) = 1$, for $\Omega > 1$. This proves $\Psi > 1$ and results in $\xi(x) > 1$.

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Supplemental material

This material is supplemental to the paper “Multiple Steady States under the Balanced Budget Rule—a Generalization”.

2.3 Dynamics

Here, we explain step by step how we can reach \dot{k}_t/k_t , that is, (9*)¹² on page 8.

At this stage, we have already established H_t and \dot{H}_t/H_t in terms of (4*) and (8*), respectively.

When we rewrite them, they are expressed as follows:

$$H_t = \left[\frac{c_t}{(1 - \tau_t) w_t} \right]^{\frac{1}{\chi}},$$

$$\frac{\dot{H}_t}{H_t} = \frac{1}{(1 - \chi) \tau_t + \chi} \left(\frac{\dot{c}_t}{c_t} (1 - \tau_t) - \frac{\dot{w}_t}{w_t} \right).$$

Here, we intend to express \dot{k}_t/k_t , which shows a part of our dynamic system.

First, we have described \dot{k}_t in (5*) as

$$\dot{k}_t = u_t \frac{K_t}{H_t} + (1 - \tau_t) w_t - \frac{c_t}{H_t} - k_t \frac{\dot{H}_t}{H_t}.$$

Substituting H_t and \dot{H}_t/H_t into the above relationship, we can express it as follows:

$$\dot{k}_t = u_t k_t + (1 - \tau_t) w_t - c_t \times \left[\frac{c_t}{(1 - \tau_t) w_t} \right]^{-\frac{1}{\chi}} - k_t \times \frac{1}{(1 - \chi) \tau_t + \chi} \left(\frac{\dot{c}_t}{c_t} (1 - \tau_t) - \frac{\dot{w}_t}{w_t} \right),$$

¹²Numbers with * regarding mathematical equations in the body all refer to those in the original manuscript, “Multiple Steady States under the Balanced Budget Rule—a Generalization.”

which yields

$$\begin{aligned}
\dot{k}_t &= \alpha k_t^{-(1-\alpha)} k_t + (1 - \tau_t) w_t - c_t^{1-\frac{1}{\chi}} (1 - \tau_t)^{\frac{1}{\chi}} w_t^{\frac{1}{\chi}} \\
&\quad - \frac{1 - \tau_t}{(1 - \chi)\tau_t + \chi} k_t \left(\frac{\dot{c}_t}{c_t} \right) + \frac{k_t}{(1 - \chi)\tau_t + \chi} \left(\frac{\dot{w}_t}{w_t} \right) \\
&= \alpha k_t^\alpha + (1 - \tau_t)(1 - \alpha) k_t^\alpha - c_t^{1-\frac{1}{\chi}} (1 - \tau_t)^{\frac{1}{\chi}} (1 - \alpha)^{\frac{1}{\chi}} k_t^{\frac{\alpha}{\chi}} \\
&\quad - \frac{1 - \tau_t}{(1 - \chi)\tau_t + \chi} k_t (\alpha k_t^{-(1-\alpha)} - \rho) + \frac{k_t}{(1 - \chi)\tau_t + \chi} \alpha \frac{\dot{k}_t}{k_t}.
\end{aligned}$$

We should describe optimal conditions for the firm as

$$\begin{cases} w_t = \eta(k_t), \\ \eta(k_t) \stackrel{\text{def}}{=} (1 - \alpha)k_t^\alpha, \end{cases}$$

and

$$\begin{cases} u_t = \phi(k_t), \\ \phi(k_t) \stackrel{\text{def}}{=} \alpha k_t^{\alpha-1}. \end{cases}$$

However, we omit such expressions for simplicity. Hereafter, w_t and u_t are both equilibrium prices.

In the above procedures, we utilise the following:

$$\begin{cases} c_t \left[\frac{c_t}{(1 - \tau_t) w_t} \right]^{-\frac{1}{\chi}} = c_t^{1-\frac{1}{\chi}} [(1 - \tau_t) w_t]^{\frac{1}{\chi}}, \\ \frac{\dot{w}_t}{w_t} = \alpha \frac{\dot{k}_t}{k_t}, \\ w_t = (1 - \alpha) k_t^\alpha, \\ \frac{\dot{c}_t}{c_t} = u_t - \rho = \alpha k_t^{-(1-\alpha)} - \rho. \end{cases}$$

Then, transforming the above formula, beginning with \dot{k} , we can reach (9*).

3.1 Two equilibria

Here, we intend to deduce a relationship, in which solutions satisfy a necessary condition for the existence of multiple equilibria.

First, by making $\dot{\lambda}_t = 0$ in (14*), we obtain

$$\begin{aligned} & [(\alpha+(1-\alpha)(1-e^{\eta^*}))((1-\chi)e^{\eta^*} + \chi) - \alpha(1-e^{\eta^*})] \left(\frac{\rho}{\alpha}\right) \\ & - [e^{\frac{\chi-1}{\chi}\delta^*} (1-e^{\eta^*})^{\frac{1}{\chi}} (1-\alpha)^{\frac{1}{\chi}} ((1-\chi)e^{\eta^*} + \chi)] \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha-\chi}{(1-\alpha)\chi}} \\ & + \rho(1-e^{\eta^*}) = 0. \end{aligned}$$

Second, in a similar fashion, by making $\dot{\eta}_t = 0$ in (15*), the following holds:

$$\begin{aligned} & [\alpha(\chi+(1-\chi)e^{\eta^*})((\alpha-\chi) - (1-\chi)(1-\alpha)e^{\eta^*})] \left(\frac{\rho}{\alpha}\right) \\ & - [\alpha(1-\alpha)^{\frac{1}{\chi}} (1-\chi)e^{\frac{\chi-1}{\chi}\delta^*} (1-e^{\eta^*})^{\frac{1}{\chi}} ((1-\chi)e^{\eta^*} + \chi)] \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha-\chi}{(1-\alpha)\chi}} \\ & + \rho(1-\alpha)((1-\chi)e^{\eta^*} + \chi) = 0. \end{aligned}$$

Additionally, we utilise the following relationships in the above processes:

$$\begin{cases} x^{\frac{1}{\chi}} \left[\frac{x}{(1-x)\chi} \right]^{\frac{\chi-1}{\chi}} = x(1-x)^{1-\chi}, \\ (1-\alpha)^{\frac{1}{\chi}} e^{\frac{\alpha-\chi}{\chi}\lambda^*} [G^\chi (w^*)^{-(\chi-1)}]^{\frac{\chi-1}{\chi}} = G^{\chi-1} (1-\alpha)^{2-\chi} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(2-\chi)-1}{1-\alpha}}, \end{cases}$$

and

$$\begin{cases} w^* = (1-\alpha) \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}, \\ e^{\lambda^*} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}. \end{cases}$$

At this stage, we define the above as $E = G^{\chi-1} (1-\alpha)^{2-\chi} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(2-\chi)-1}{1-\alpha}}$.

Here, consider the function $h(x)$, which is defined (24*):

$$\begin{aligned} h(x) &= (1-\alpha)(\chi-1)x^2 + (\alpha\chi + 1 - 2\alpha)x + \alpha \\ & - \left(\frac{\alpha}{\rho}\right) E((\chi-1)x + 1)(1-x)^{1-\chi} x, \end{aligned}$$

which can be transformed as

$$h(x) = (\chi-1)\left(x - \frac{1}{1-\chi}\right) \left[(1-\alpha) \left(x + \frac{\alpha}{1-\alpha}\right) - \left(\frac{\alpha}{\rho}\right) E(1-x)^{1-\chi} x \right].$$

From what we have analysed so far, we must explore our solution in conditions which satisfy $\phi(x) = 0$. We define $\phi(x)$ as

$$\begin{aligned}\phi(x) &= (1 - \alpha) \left(x + \frac{\alpha}{1 - \alpha} \right) - \left(\frac{\alpha}{\rho} \right) E (1 - x)^{1-\chi} x \\ &= (1 - \alpha)\varphi(x).\end{aligned}$$

Here, we newly define a function $\varphi(x)$ as

$$\varphi(x) = \left(x + \frac{\alpha}{1 - \alpha} \right) - (1 - \alpha)^{1-\chi} G^{\chi-1} \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha(1-\chi)}{1-\alpha}} (1 - x)^{1-\chi} x.$$

At this stage, we again define B as

$$B = (1 - \alpha)^{1-\chi} G^{\chi-1} \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha(1-\chi)}{1-\alpha}}.$$

In this situation, $\varphi(x)$ can be expressed as

$$\begin{aligned}\varphi(x) &= x + \frac{\alpha}{1 - \alpha} - B(1 - x)^{1-\chi} x \\ &= x[1 - B(1 - x)^{1-\chi}] + \frac{\alpha}{1 - \alpha}.\end{aligned}$$

From what we have analysed, we finally obtain $\phi(x) = (1 - \alpha)\varphi(x)$. This shows that solutions satisfying $\varphi(x) = 0$ are those in the steady states which we are exploring.

3.2 Maximum G

The following shows how τ^* , c^* , H^* and G are determined, respectively. Of course, τ^* , c^* and H^* are the values corresponding to G .

First, since $\tau^* = 1 - x^*$,

$$\tau^* = \frac{\chi(2 - \alpha) - 2 + \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1 - \alpha)(\chi - 1)}.$$

Second, we explore the value of c^* .

Based on the previous definition (19*), we obtain:

$$q = (e^{\delta^*})^{\frac{\chi-1}{x}} = (c^*)^{\frac{\chi-1}{x}}.$$

Furthermore, considering (21*),

$$q = \left[\frac{G^\chi}{(w^*)^{\chi-1}} \frac{x}{(1-x)\chi} \right]^{\frac{\chi-1}{\chi}},$$

we obtain the following required relationship :

$$c^* = q^{\frac{\chi}{\chi-1}} = \frac{G^\chi(1-\tau^*)}{(w^*)^{\chi-1}(\tau^*)^\chi}.$$

Here, G is formulated as follows:

$$G = (1-\alpha) \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha}{1-\alpha}} \left[\frac{(\tau^*)^\chi}{1-(2-\chi)(1-\tau^*)} \right]^{-\frac{1}{1-\chi}}.$$

Utilising this G , we eventually reach:

$$c^* = (1-\alpha) \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha}{1-\alpha}} (1-\tau^*) \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)} \right]^{-\frac{\chi}{1-\chi}}.$$

Third, we explore H^* . From (4*) we can express H^* as follows:

$$H^* = \left[\frac{c^*}{(1-\tau^*) w^*} \right]^{\frac{1}{\chi}} = (c^*)^{\frac{1}{\chi}} (1-\tau^*)^{-\frac{1}{\chi}} (w^*)^{-\frac{1}{\chi}}.$$

Thus, substituting c^* and $w^* = (1-\alpha) \left(\frac{\alpha}{\rho} \right)^{\frac{\alpha}{1-\alpha}}$ into the above relationship, we obtain the following, which we investigate:

$$H^* = \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)} \right]^{-\frac{1}{1-\chi}}.$$