Alternative Resolution to the Mehra-Prescott Puzzle: Verification by the Original Data

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Alternative Resolution to the Mehra–Prescott Puzzle: Verification by the Original Data

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Abstract

Many extensive debates followed Mehra and Prescott's (1985) sensational empirical results concerning the equity premium embodied in household equity portfolios. The problem of the equity premium—the Mehra–Prescott puzzle—arises because researchers overlook the factor of uncertainty in household consumption behaviour, thereby failing to account for the offsetting effect in the intertemporal substitution of consumption. Although many US empirical studies reject the consumption-based capital asset pricing model under a time-separable constant relative risk aversiontype utility function, we resolve this problem by formulating an expandedEuler equation that accommodates uncertainty using Mehra–Prescott's original data.

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1. Introduction

Empirical research to validate the consumption-based capital asset pricing model (C-CAPM) has been developed since the 1970s and has become a litmus test of economic theory. Regardless of refinements that extend data (diversifying the consumption series or asset categories), change sample periods or alter forms of function, the standard preference structure still cannot sufficiently explain the interaction of consumption and equity returns. One example of this existing issue is a contradiction called 'equity premium puzzle', reported by Mehra and Prescott (1985). It shows that theoretical values in annual samples of US stock index returns, returns on Treasury notes and growth rates for consumption from 1890 to 1979 under a time separable Constant Relative Risk Aversion (CRRA)-type utility function explain only a fraction of the excess return provided by the average rate of return (ROR) of stocks price index over the average return on Treasury notes (equity premium).

We theoretically and empirically resolve this puzzle using a standard CRRA-type utility function and the data used by Mehra and Prescott (1985). We formulate an Euler equation (hereinafter referred to as 'the expandedEuler equation') which includes uncertainty as a variable to explain household consumption. Our model (hereinafter 'the uncertainty model') is derived from the expected utility function under uncertainty based on the precautionary saving theory. Further, we refer to the conventional model derived from the utility function under certainty as 'the certainty model'.

This paper is constructed as follows. Section 2 surveys previous solutions to the puzzle and outlines our resolution. Section 3 derives the expanded Euler equation for consumption under uncertainty and formulates it as a measurable form under specified assumptions. It derives a decision level of the degree of relative risk aversion in our model (uncertainty model) by extending Mankiw and Zeldes's (1991) analytical model. Section 4 shows that the uncertainty model resolves the equity premium puzzle by computing the degree of relative risk aversion in the certainty and uncertainty models and comparing results using Mehra and Prescott's (1985) data and an uncertainty index. Section 5 considers the economic implications of the uncertainty term in the uncertainty model. Section 6 summarises and concludes.[1]

2. Literature review and an alternative resolution

Many studies have proposed solutions to the equity premium puzzle posed by Mehra and Prescott (2003). Generally, however, solutions have been limited to introducing an alternative preference structure (time non-separable and habit formation). Epstein and Zin (1989, 1991) propose a representative time non-separable model. Constantinides (1990) proposes an internal habit formation model. Campbell and Cochrane (1995) incorporate the prospect of a recession into differential type internal habit model. Abel (1990) also proposes external habit model. Some alternative preference structures have successfully calculated reasonable levels of relative risk aversion, but none has theoretically and empirically resolved the puzzle using a standard CRRA-type utility function.

Mankiw and Zeldes (1991) present an analytical model to clarify this puzzle. They show that the degree of relative risk aversion is 26.3 using Mehra and Prescott's (1985) annual data spanning 1890–1979, but it becomes 89.0 using the annual data for the post-war period 1948–1988. Both results far exceed the empirically conceivable range of relative risk aversion (10 or less). They point out that the puzzle arises because consumption growth covaries too little with the return on equitiesto justify the large risk premium in equity returns.

The expected ROR on assets based on the Euler equation for consumption under the CRRA-type utility function is expressed as follows. It is the concept of the C-CAPMthat a big risk premium is requested about the assets which are not useful for

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[1] In the case of specifying the CRRA-type preference ($\rho = \gamma$) under the Kreps–Porteus type preference and non-iid dividend growth process, Weil (1989) pointed out that a risk-free interest rate to get the real risk premium level became unusually high. This contradiction is known as the risk-free rate puzzle. Although our resolution to the equity premium puzzle also resolves the risk-free rate puzzle, we address the latter in an upcoming study.

levelling consumption (or assets wherein the covariance of numerators between the stochastic discount factor and ROR on assets is a large negative value).

$$E_t[(1+r_{jt+1})] = \left\{1 - \mathrm{cov}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, (1+r_{jt+1})\right]\right\} \Big/ E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] \quad \text{(j=1,2,...,N)}$$

Here, if a growth rate for consumption covaries too little with the return on equities, the covariance in numerators between the stochastic discount factor and the ROR on assets does not become large and negative, as expected. The large risk premium cannot be justified except by reducing the denominator (i.e. increasing the relative risk aversion exponentially).

However, when households consider uncertainty in making consumption decisions, situations wherein the ROR on financial assets (equity premium) rises coincide with economic upturns during which uncertainty diminish and a reversal of precautionary saving accelerates future consumption into the present. Therefore, a substitution toward the future in consumption by higher ROR on financial assets will be offset. Conversely, declines in the ROR on financial assets (the equity premium) coincide with a cooling economy. Uncertainty increases, and precautionary saving also accordingly rises. Current consumption is postponed. Therefore, a substitution toward the present in consumption by lower ROR on financial assets will be offset. These offsetting effects theoretically explain the large risk premium on equities when covariance between growth in consumption and the ROR on assets is small.

In addition these offsets suggest that the equity premium puzzle arises because uncertainty is excluded as a variable in an Euler equation for consumption. Thus, our analysis theoretically and empirically resolves the equity premium puzzle by acknowledging precautionary savings and inserting an uncertainty variable into an Euler equation for consumption.

Skinner (1988) shows that expected marginal utility when the future is uncertain equals marginal utility when uncertainty is absent multiplied by $1+0.5(\gamma+\gamma^2)\sigma_w^2$ (hereinafter the 'uncertainty premium ratio') under a CRRA-type utility function. γ denotes the degree of relative risk aversion. σ_w^2 is the squared value of the coefficient of variation. ($\sigma_w^2 = Var(W)/(\overline{W})^2$: $\overline{W}$ is the expected value of financial assets.) [2] Skinner's (1988) uncertainty premium ratio includes an uncertainty variable and indicates the relevant procedure for addressing the equity premium puzzle. We must formulate an Euler equation that acknowledges shifting marginal utility for consumption. Doing so requires estimating by two explanatory variables: conventional ROR on financial assets and uncertainty for households. The estimation must consider the offsetting effects of accelerated (postponed) consumption generated by reduced (heightened) economic uncertainty. The result will resolve the apparent misalignment in the risk premium puzzle.

3. Model

3.1 Derivation of the expanded Euler equation for consumption under uncertainty

We assume that consumption becomes uncertain as households estimate future changes in employment and income. That degree of uncertainty is expressed by fluctuation in the arithmetic means of consumption data. In this case, a household's expected marginal utility for consumption given uncertainty is expressed in the following form, which includes an uncertainty premium ratio.[3]

$$U^{*}{}'(C_t) = C_t^{-\gamma}\left[1 + 0.5(\gamma + \gamma^2)CV_t^2 - \gamma h'(C_t)CV_t\right] \quad (1)$$

C_t represents the mean value of real consumption at period t. CV_t^2 represents the square of the degree of uncertainty surrounding consumption at period t: $CV_t^2 = (h_t / C_t)^2$ where $h_t = h(C_t)$ ($h'(C_t) > 0$) expresses the range of fluctuation that the economy imposes on consumption through its influence on assets. γ represents a constant degree of relative risk aversion.[4]

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2 The magnitude of a risk premium per unit of financial assets can be referred to the calculation result of $(\overline{W} - \hat{W})/\overline{W}$ by Eq. (6) in Skinner (1988, p.241). In addition, an uncertainty premium ratio can be referred to in Eq. (7) in Skinner (1988, p.241) and its disclaimer.

3 Details of the derivation process are available upon request.

4 Here, C_t expresses the mean $(C_t^A + C_t^B)/2$ of the two consumption values (C_t^A, C_t^B) under a probability of 50% respectively caused by uncertainty as seen in Figure 7.3 of Romer (1996) Chapter 7. h_t expresses fluctuation range to each

From Equation (1), expected marginal utility under uncertainty is the expected marginal utility in the certainty model multiplied by an uncertainty premium ratio $[1+0.5(\gamma+\gamma^2)CV_t^2-\gamma h'(C_t)CV_t]$. The second term expresses the precautionary saving effect attributable to heightened uncertainty. The third term expresses that the increase in uncertainty reduces precautionary savingeffect according to the size of $h'(C_t)$.

The intertemporal optimal consumption model that uses the household's expected utility function under uncertainty is set as follows:

$$\max\ E_t[\sum_{i=0}^{\infty}\beta^i U^*(C_{t+i})] \tag{2}$$

$$\text{s.t.}\ \sum_{j=1}^{N} q_{jt}A_{jt+1}+C_t=\sum_{j=1}^{N}(q_{jt}+d_{jt})A_{jt}+Y_t \tag{3}$$

β is the subjective discount rate $(0<\beta<1)$. q_{jt} is the price of asset j at period t (j=1,2,…,N). d_{jt} is the dividend derived from asset j at period t (j=1,2,…,N). A_{jt} is the value of asset j at period t. Y_t is non-asset income for period t. $E_t[\bullet]$ is the conditional expectation operator based on information available at time t.

Solving the above optimization problem yields the following first-order condition for maximization:

$$E_t[\beta\frac{U^{*\prime}(C_{t+1})}{U^{*\prime}(C_t)}(\frac{q_{jt+1}+d_{jt+1}}{q_{jt}})]-1=0. \tag{4}$$

ROR r_{jt+1} on asset j is defined as $r_{jt+1}=(q_{jt+1}+d_{jt+1})/q_{jt}-1$ so that $(q_{jt+1}+d_{jt+1})/q_{jt}$ in Equation (4) can be replaced by $(1+r_{jt+1})$. Given this replacement and by substituting Equation (1) into Equation (4), the household's expanded Euler equation for consumption under uncertainty in the CRRA-type utility function can be expressed as

$$E_t[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{1+0.5(\gamma+\gamma^2)CV_{t+1}^2-\gamma h'(C_{t+1})CV_{t+1}}{1+0.5(\gamma+\gamma^2)CV_t^2-\gamma h'(C_t)CV_t}(1+r_{jt+1})]-1=0 \quad \text{(j=1,2,…,N.} \tag{5}$$

In considering the relation between h_t and C_t to judge the impact of $h'(C_t)$ and

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C_t^A ・ C_t^B from this mean value as the degree of uncertainty of consumption.

$h^{'}(C_{t+1})$ on Equation (5), h_t express the range of fluctuation imposed by economic variables on consumption through their influence on total assets. C_t expresses the mean $(C_t^A + C_t^B)/2$ of consumption C_t^A and C_t^B under uncertainty. Therefore, it will be the relationship between the arithmetic mean of consumption C_t, which rises gently, and fluctuation range h_t, which fluctuates intensively although correlated to C_t. A scatter plot for both can be constructed from actual data. The apparently random scatter of data can be clarified by regression, and the coefficients obtained will be close to 0 (i.e. $h^{'}(C_t) \cong 0$).

We separate the entire period of US per capita consumption data (1890–1979) presented by Mehra and Prescott (1985) into cycles and trends using a Hodrick–Prescott filter. We estimate $h_t = \alpha + \beta C_t + u_t$, where the absolute value of a cycle component is an explained variable and the trend component is an explanatory variable. Then, $\beta = dh_t / dC_t$ exhibits a much smaller value.[5]

Following the discussion above, we proceed with the analysis under the assumption that $h_t = h(C_t)$ is subject to $h^{'}(C_t) > 0$ and simultaneously subject to $h^{'}(C_t) \cong 0$. Under this assumption, the uncertainty premium ratio in Equation (1) becomes $1 + 0.5(\gamma + \gamma^2)CV_t^2$ and is expressed in a manner similar to that presented by Skinner (1988). Transforming the equation using a first-order approximation of the exponential function of the Taylor expansion formula yields $1 + 0.5(\gamma + \gamma^2)CV_t^2 \cong \exp[0.5(\gamma + \gamma^2)CV_t^2]$. Therefore, the middle term of Equation (5) can be transformed as

$$\frac{1 + 0.5(\gamma + \gamma^2)CV_{t+1}^2}{1 + 0.5(\gamma + \gamma^2)CV_t^2} \cong \left(\frac{\exp(CV_{t+1}^2)}{\exp(CV_t^2)} \right)^{0.5(\gamma + \gamma^2)}.$$

Applying the transformed middle term to Equation (5) leads to the following expanded Euler equation for consumption. Doing so also adds the growth rate of the exponential of the squared value of the degree of uncertainty surrounding consumption as an explanatory variable. The coefficient $0.5(\gamma + \gamma^2)$—the composite of the degree of relative risk aversion—is applied as an exponent of that growth rate in the degree of

[5] More specifically, the estimated value became −0.0033 (0.0046) (value in parenthesis is the standard error of the estimate).

uncertainty for consumption.

$$\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{\exp(CV_{t+1}^2)}{\exp(CV_t^2)}\right)^{0.5(\gamma+\gamma^2)}(1+r_{jt+1})=1 \qquad (6)$$

We analyse this expanded Euler equation for consumption under uncertainty as a measurement object formalized by three variables—the consumption growth rate, ROR on assets and growth rate of the degree of uncertainty surrounding consumption—as explanatory variables.

3.2 Expanding Mankiw and Zeldes (1991) to a three-variable model

Mankiw and Zeldes (1991) apply the Taylor expansion of the two variables functions to the Euler equation (7) of two explanatory variables (consumption growth rate and ROR on assets) and derive the relational expression (8) under some omissions among the equity premium, the degree of relative risk aversion and the covariance between the ROR on assets and consumption growth.

$$E[(1+r^i)(1+g^C)^{-\gamma}]=1+\rho \qquad (7)$$

$$E[r^i]-\bar{r}\cong\gamma Cov(r^i,g^C) \qquad (8)$$

$g^C=(C_{t+1}/C_t)-1$, and the time subscript is omitted. r^i represents the ROR on risky asset i. $\bar{r}$ represents the ROR on the risk-free asset. $E[r^i]-\bar{r}$ represents the equity premium. ρ represents the time preference rate (equivalent to $(1/\beta)-1$).

In accord with Equation (8), the degree of relative risk aversion in the certainty model is defined as the equity premium divided by the covariance between the ROR on assets and consumption growth. That is,

$$\gamma_{CM}^*\cong(E[r^i]-\bar{r})/\sigma_{ic}. \qquad (9)$$

$\sigma_{ic}=Cov(r^i,g^C)$, and the subscript on γ expresses an abbreviation of the certainty model (CM).

The uncertainty model formalised by Equation (6) can be expressed in the manner of Equation (7) for the certainty model as follows.

$$E[(1+r^i)(1+g^C)^{-\gamma}(1+g^{eCVSQ})^{0.5(\gamma+\gamma^2)}]=1+\rho. \qquad (10)$$

$g^{eCVSQ}=(\exp(CV_{t+1}^2)/\exp(CV_t^2))-1$, and the time subscript is omitted.

Applying the Taylor expansion of the three variables functions to Equation (10) and calculating in the manner of Mankiw and Zeldes (1991) under some omissionsleads to the following equation. It relates the equity premium, degree of relative risk aversion and covariance between the ROR on assets and the consumption growth rate, as well as the rate at which the degree of uncertainty grows:[6]

$$E[r^i]-\bar{r} \cong \gamma Cov(r^i,g^C)-0.5(\gamma+\gamma^2)Cov(r^i,g^{eCVSQ}). \tag{11}$$

Applying $Cov(r^i,g^{eCVSQ})=0$ to Equation (11) yields the equation for determining the degree of relative risk aversion in the certainty model (Equation (8)).

By solving equation (11) for γ, the decision level of the degree of relative risk aversion in the uncertainty model can be defined as follows:

$$\gamma^*_{UCM} \cong \frac{2(\sigma_{ic}-0.5\sigma_{iv})\pm\sqrt{4(\sigma_{ic}-0.5\sigma_{iv})^2-8\sigma_{iv}(E[r^i]-\bar{r})}}{2\sigma_{iv}}. \tag{12}$$

$\sigma_{ic}=Cov(r^i,g^C)$, $\sigma_{iv}=Cov(r^i,g^{eCVSQ})$ and the subscript on γ expresses the abbreviation of the uncertainty model (UCM).

The condition for solving the degree of relative risk aversion given a positive risk premium by the discriminant $D=4(\sigma_{ic}-0.5\sigma_{iv})^2-8\sigma_{iv}(E[r_i]-\bar{r})$ from Equation (12) is $\sigma_{iv}=Cov(r^i,g^{eCVSQ})\leq 0$. Therefore, the appropriate choice of an uncertainty index in which uncertainty recedes (rises) when ROR on assets rises (falls) is required to solve the uncertainty model.

4. Empirical analysis

4.1 Data

Data for calculating the degree of relative risk aversion comes from the annual sample spanning 1889–1979 (excluding 1889) used by Mehra and Prescott (1985). It consists of the following series.[7]

(i) Series P: Annual average Standard & Poor's (S&P) Composite Stock Price Index divided by the consumption deflator

(ii) Series D: Real annual dividends for the S&P series

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6 Details of the derivation process are available upon request.

7 Data have been published at Academic Web Pages by Rajnish Mehra. Available: http://www.academicwebpages.com/preview/mehra/resources/ (accessed 12 March 2016)

(iii) Series RF: Annual average nominal return on three-month Treasury bills

(iv) Series PC: Consumption deflator series

(v) Series C: Per capita consumption of nondurables and services in thousands of 1972 dollars

We use the annual average US unemployment rate (unrate) spanning 1890–1979 published since 1890 as data for the uncertainty index (hereinafter ‘the unrate sequence’).[8]

4.2 Processing methods

According to the method described in Section 2 of Mehra and Prescott’s (1985) paper, we first calculate the average annual real ROR on equity using Series P and Series D. We calculate the consumption growth rate using Series C and the real ROR on risk-free securities using Series RF and PC.

Second, we calculate the risk premium (RP) as the difference between the real ROR on equity and the real ROR on risk-free securities. Descriptive statistics of these variables appear in the upper part of Table 1 (1890–1978). Means and standard deviations are almost identical to those in Table 1 (1889–1978) presented by Mehra and Prescott (1985, p.147).

There are no data that directly describe the degree of uncertainty surrounding consumption in the United States. Therefore, we consider three patterns of cases wherein the mean (scale) of the uncertainty index before calculating the growth rate is 0.1, 0.3, and 0.5.[9] The method of data processing is described below.

First, to secure stationarity of data, we set a Hodrick–Prescott filter (λ = 14400) to

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8 The source of the data is as follows.: 1890 to 1970; Historical Statistics of the United States Colonial Times to 1970 (U.S. Department of Commerce), 1971 to 1979; Labor Force Statistics from the Current Population Survey(U.S. Department of Labor)

9 A related index, the coefficient of variation of income, which shows the degree of uncertainty surrounding income, appears in *OECD Regions at a Glance 2016*. Available: http://www.oecd-ilibrary.org/governance/oecd-regions-at-a-glance-2016_reg_glance-2016-en (accessed 28 August 2016) According to this source, the coefficient of the variation in US disposable income was 0.13 in 1995 and 0.16 in 2014.

the unrate sequence and the sequence that consists only of cycles remaining after the trend is extracted (hereinafter 'the unrate_c sequence').

Next, we calculate arithmetic means of the unrate sequence from 1890 through 1979 and add the unrate_c sequence to create a steady sequence for an uncertainty index (hereinafter 'the unrate_s sequence'). After calculating mean values of the unrate_s sequence, which accords with the mean value of the unrate sequence, we convert unrate_s to a simple magnification-adjusted sequence so its average values become 0.1, 0.3 and 0.5 (hereinafter 'the CV sequence').[10]

Finally, we calculate $\exp(CV_{t+1}^2)/\exp(CV_t^2)$ from the CV sequence to construct the growth sequence for the uncertainty index for each case (0.1, 0.3 and 0.5) (hereinafter 'the gecvsq sequence').

The transition of the uncertainty index data and the growth rate of the uncertainty index for each case appear in Figures 1 and 2. Shadows on Figure 1 indicate recessions (except the first year, which recovers from the valley).

【 Figure 1 】

【 Figure 2 】

4.3 Estimation results

Two cases of descriptive statistics for data used to calculate the degree of relative risk aversion are presented in Table 1. The first spans 1890–1978, the period Mehra and Prescott (1985) examine, and the second pertains to the post-war period (1946–1978), and the same applies for subsequent tables.[11]

【 Table 1 】

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[10] Magnification of the adjustment becomes 0.1 (or 0.3 or 0.5)/the mean value of unrate_s, respectively.

[11] The telophase of the subsequent data period is set to the final year of the data after calculating the growth rate.

In Table 1, *cons* denotes the real consumption growth rate plus one, *stocks* the real ROR on the stock price index plus one and *bills* the real ROR on Treasury securities (risk-free assets) plus one. *RP* denotes the risk premium and *unrate* (mean = 0.1, 0.3, 0.5) represents the growth rate of the uncertainty index plus one when its mean values created from the unemployment rate are 0.1, 0.3 and 0.5. From Table 1, the mean for the risk premium for the entire period (post-war period) is 6.284% (7.333%).

Before calculating the degree of relative risk aversion based on Equations (9) and (12), Table 2 shows calculations of the variance and covariance matrix which contains the covariance between ROR on equity and the consumption growth rate and between ROR on equity and the growth rate of the uncertainty index.

【 Table 2 】

For the entire period (1890–1978), covariance between the ROR on equity and the consumption growth rate is 0.002212. Covariance between the ROR on equity and the growth rate of the uncertainty index are negative values between −0.000537 and −0.015507. The risk premium is a positive value (6.284%). Therefore, the discriminant condition in Equation (12) is fulfilled.

During the post-war period (1946-1978) covariance between ROR on equity and the consumption growth rate is positive (0.000547). Covariance between ROR on equity and the growth rate of the uncertainty index is a negative value between −0.000080 and −0.001847. The risk premium is positive (7.333%). The discriminant condition of in Equation (12) is also fulfilled.

After substituting these covariance and risk premiums into Equations (9) and (12), estimation results for the degree of relative risk aversion are as shown in Table 3.

【 Table 3 】

However, CM represents estimation results of the certainty model based on Equation (9), and UCM (unrate) (mean = 0.1, 0.3, 0.5) represents estimation results of the uncertainty model based on Equation (12) when mean values of the uncertainty index are set to 0.1, 0.3 and 0.5.

From the table, the degree of relative risk aversion calculated by the certainty model for the entire period (1890−1978) becomes 28.4. For the post-war period (1946−1978), it becomes 134.1. For the entire period (1890−1978), it is almost the same as the estimation result (26.3) in Table 3 (1890-1979) presented by Mankiw and Zeldes (1991, p.104). However, during the post-war period (1946−1978), its value is greater than the estimation result (89.0) in that table (1948-1988), primarily because periods differ. Both values become the excess value identified in the equity premium puzzle.

In the uncertainty model, for the entire period (1890−1978), the degree of relative risk aversionbecomes 11.4 when the mean value of the uncertainty index is 0.1, 4.1 when it is 0.3, and 2.3 when it is 0.5. During the post-war period (1946−1978) it becomes 36.0 when the mean value of the uncertainty index is 0.1, 13.2 when it is 0.3, and 8.1 when it is 0.5. The estimation result has improved dramatically. When the mean value of the uncertainty index is 0.5, the degrees of relative risk aversion in the partial and entire periods are calculated within 10. These results indicate that the uncertainty model may resolve the equity premium puzzle.

5. Discussion

For considering the economic meaning of the uncertainty term (third item on the left side of Equation (6)), we confirm the derivation process of the indifference curve under uncertainty from a two-period expected utility function extracted from a multi-period model by using the four-quadrant diagram in Figure 3. However, $\rho(C_t, h_t)$ represents the width of decline of utility in C_t caused by uncertainty. When there are no uncertainties in both periods ($h_{t+1} = h_t = 0$, i.e., $\rho(C_{t+1}, h_{t+1}) = \rho(C_t, h_t) = 0$), household's expected utility functions parallel the CRRA-type utility functions under certainty. Accordingly, an indifference curve will parallel that in the certainty model.

【 Figure 3 】

For calculating the marginal rate of substitution (MRS) of the indifference curve in the uncertainty model, extracting utility function only for periods I = 0, 1 in Equation (2) and setting total utility as Z yields Equation (13).

$$Z = U^*(C_t) + \beta U^*(C_{t+1}) \tag{13}$$

The expanded Euler equation (Equation (6)) indicates the first-order conditions of utility maximization derived from the optimum consumption model for multiple periods. The conditions mean that the intertemporal MRS of indifference curve $-dC_{t+1}/dC_t$ matches the inclination $(1+r_{t+1})$ of the budget constraint line. To determine this MRS, conducting total differentiation against equation (13) and setting it to 0 yields the following equation from $dZ = U^{*}{}'(C_t)dC_t + \beta U^{*}{}'(C_{t+1})dC_{t+1} = 0$: [12]

$$-\left.\frac{dC_{t+1}}{dC_t}\right|_{dZ=0} = \frac{U^{*}{}'(C_t)}{\beta U^{*}{}'(C_{t+1})} = \frac{C_t^{-\gamma}[1+0.5(\gamma+\gamma^2)CV_t^2]}{\beta C_{t+1}^{-\gamma}[1+0.5(\gamma+\gamma^2)CV_{t+1}^2]}. \tag{14}$$

By incorporating deformation using the Taylor expansion formula of exponential functions against the uncertainty term in Equation (14), as mentioned in Subsection 3.1, the MRS of the indifference curve in the multi-period uncertainty model can be expressed using the growth rate of the uncertainty index as follows:[13]

$$-\left.\frac{dC_{t+1}}{dC_t}\right|_{dZ=0} = \frac{1}{\beta}\left(\frac{C_t}{C_{t+1}}\right)^{-\gamma}\left(\frac{\exp(CV_t^2)}{\exp(CV_{t+1}^2)}\right)^{0.5(\gamma+\gamma^2)} \tag{15}$$

On the other hand, since the MRS of the indifference curve in the multi-period model of the certainty model is obtained by setting the degree of uncertainty surrounding consumption CV_t in Equation (15) to 0, it can be expressed by the following equation in which the uncertainty term is deleted by substituting $\exp(0)=1$ into the numerator and denominator of Equation (15).

[12] In accordance with Section 3.1, we insert the assumption $h'(C_t) \cong 0$ in the formula for the uncertainty premium ratio.

[13] Setting the MRS in Equation (15) equal to the tilt $(1+r_{t+1})$ of the budget constraint line and transforming it yield the expanded Euler equation for consumption under uncertainty (Equation (6)).

$$-\left.\frac{dC_{t+1}}{dC_t}\right|_{dZ=0}=\frac{1}{\beta}\left(\frac{C_t}{C_{t+1}}\right)^{-\gamma} \tag{16}$$

If uncertainty rises from period t to period t+1 because of an economic downturn and $CV_{t+1}>CV_t$, the MRS of the indifference curve in the certainty model derived from Equation (16) remains unchanged. However, the growth rate of the uncertainty index $\exp(CV_t^2)/\exp(CV_{t+1}^2)$ in the third item on the right side of Equation (15) falls below 1 in the uncertainty model. Thus, the MRS of the entire indifference curve declines (or a change focused on future consumption occurs).

If uncertainty from period t to t+1 declines because of an economic upturn and $CV_t > CV_{t+1}$, the MRS of the indifference curve in the certainty model derived from Equation (16) remains unchanged. However, the growth rate of the uncertainty index $\exp(CV_t^2)/\exp(CV_{t+1}^2)$ at the third item on the right side of Equation (15) exceeds 1 in the uncertainty model. The MRS for the entire indifference curve increases (or a change focused on present consumption occurs).

As discussed, changes in the indifference curve that coincide with economic fluctuations offset the acceleration (postponement) of consumption caused by shifting ROR on financial assets. This creates a situation wherein the covariance between consumption growth rates and the ROR on financial assets is small when risk premiums are large. From the perspective of C-CAPM, the expected ROR on assets based on the expanded Euler equation for consumption under the CRRA-type utility function is expressed as follows:

$$E_t[(1+r_{jt+1})]=\left\{1-\operatorname{cov}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{\exp(CV_{t+1}^2)}{\exp(CV_t^2)}\right)^{0.5(\gamma+\gamma^2)},(1+r_{jt+1})\right]\right\}$$

$$\Big/ E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{\exp(CV_{t+1}^2)}{\exp(CV_t^2)}\right)^{0.5(\gamma+\gamma^2)}\right] \quad \text{(j=1,2,…,N)}$$

Compared with the definitional equation of the certainty model in Section 2, the stochastic discount factor of the numerator and denominator have changed to those caused by multiplying the uncertainty term. Further, the covariance between the growth rate of the uncertainty index (degree of uncertainty) and ROR on assets becomes a large negative value. Therefore, even if the covariance between consumption growth rates and the ROR on financial assets is small, numerators' covariance between the stochastic discount factor and ROR on assets becomes a large negative value, justifying the large risk premium. In addition, since it becomes unnecessary to make the degree of relative risk aversion abnormally large in order to reduce the denominator, the abnormal value of the relative risk aversion presented by Mankiw and Zeldes (1991) disappears.

As explained, the growth rate of the uncertainty index in the uncertainty model functions as an explanatory variable for the utility curve in determining the consumption growth rate, which aggregates the relative uncertainty between the present and future and reflects it in the MRS for the entire generalised indifference curve under uncertainty. This is based on the idea that households confronting uncertainty that may impact total assets do not behave in accord with the utility function under certainty, but they do so while adjusting consumption according to the utility function that incorporates uncertainty. The uncertainty term in the uncertainty model corrects defects in the estimation of parameters that occur under the certainty model, which disregards uncertainty, by reflecting household behaviour under uncertainty in the dynamic optimization model for consumption.

6. Conclusion

The equity premium puzzle has discouraged applying the practical aspects of C-CAPM in research for 30 years. That is largely because the dynamic optimization problem for intertemporal consumption and the issue of precautionary saving, which considers consumption behaviour under uncertainty, have been treated as separated topics.

We have unified these considerations under one expanded Euler equation and shown that the equity premium puzzle can be resolved under a CRRA-type utility function by considering the offsetting effect in the intertemporal consumption. Our approach enables the standard preference structure to describe data for consumption and equity returns adequately.

That the uncertainty model resolves the equity premium puzzle illustrates that rates of return on financial assets are macro-environmental factors to which households passively or indirectly react. In contrast, uncertainties surrounding employment and income that affect total assets are micro-environmental factors to which households react directly for optimizing consumption.

To strengthen the analytical power of the macro-economic model and regain confidence in asset pricing, future studies need to deepen the analysis of the intertemporal substitution of consumption, which considers the impact of precautionary savings.

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Table 1

Descriptive statistics (annual basis)

Time periods		cons	stocks	bliis	unrate mean=0.1	unrate mean=0.3	unrate mean=0.5	RP
1890–1978	max	1.11111	1.50983	1.17615	1.03097	1.31582	2.13639	0.54261
(N=89)	min	0.90909	0.62962	0.76507	0.97499	0.79618	0.53238	−0.47785
	mean	1.01878	1.07009	1.00726	1.00009	1.00383	1.02879	0.06284
	s.d.	0.03554	0.16539	0.05904	0.00910	0.08493	0.26481	0.16742
1946–1978	max	1.04194	1.40642	1.02196	1.00862	1.08035	1.23833	0.41395
(N=33)	min	0.97949	0.72323	0.89480	0.99356	0.94355	0.85155	−0.23286
	mean	1.01816	1.06739	0.99406	0.99996	1.00011	1.00288	0.07333
	s.d.	0.01405	0.13740	0.02633	0.00354	0.03228	0.09183	0.13141

Notes: Each variable is as follows.

cons : growth rate of per capita real consumption + 1
stocks : average annual real rate of return on equity + 1
bliis : real rate of return on risk-free security + 1
unrate : growth rate of the uncertainty index + 1
RP : risk premium

The numbers "mean = 0.1,0.3,0.5" attached to "unrate" represents the average value of the uncertainty index before calculating the growth rate (after processing for stationarity and magnification adjusting).

Table 2

Calculation of variance and covariance

Time periods		cons	stocks	bliis	unrate mean=0.1	unrate mean=0.3	unrate mean=0.5
1890–1978	cons	0.001263	0.002212	−0.000153	−0.000151	−0.001404	−0.004241
(N=89)	stocks		0.027354	0.001405	−0.000537	−0.005033	−0.015507
	bliis			0.003485	0.000220	0.002143	0.007086
	unrate (mean=0.1)				0.000083	–	–
	unrate (mean=0.3)					0.007214	–
	unrate (mean=0.5)						0.070122
1946–1978	cons	0.000197	0.000547	0.000185	−0.000027	−0.000246	−0.000697
(N=33)	stocks		0.018880	0.001152	−0.000080	−0.000704	−0.001847
	bliis			0.000693	0.000015	0.000136	0.000378
	unrate (mean=0.1)				0.000013	–	–
	unrate (mean=0.3)					0.001042	–
	unrate (mean=0.5)						0.008433

Notes: Each variable is as follows.

cons : growth rate of per capita real consumption + 1
stocks : average annual real rate of return on equity + 1
bliis : real rate of return on risk-free security + 1
unrate : growth rate of the uncertainty index + 1

The numbers "mean = 0.1,0.3,0.5" attached to "unrate" represents the average value of the uncertainty index before calculating the growth rate (after processing for stationarity and magnification adjusting).

Table 3

Estimation results of the degree of the relative risk aversion

Time periods	CM	UCM（unrate）		
		mean=0.1	mean=0.3	mean=0.5
1890–1978 (N=89)	28.4	11.4	4.1	2.3
1946–1978 (N=33)	134.1	36.0	13.2	8.1

Notes: CM represents the estimation result of "the certainty model " and UCM represents that of "the uncertainty model" respectively.

Figure 1

Transition of the uncertainty index after processing for stationarity (unrate_s)

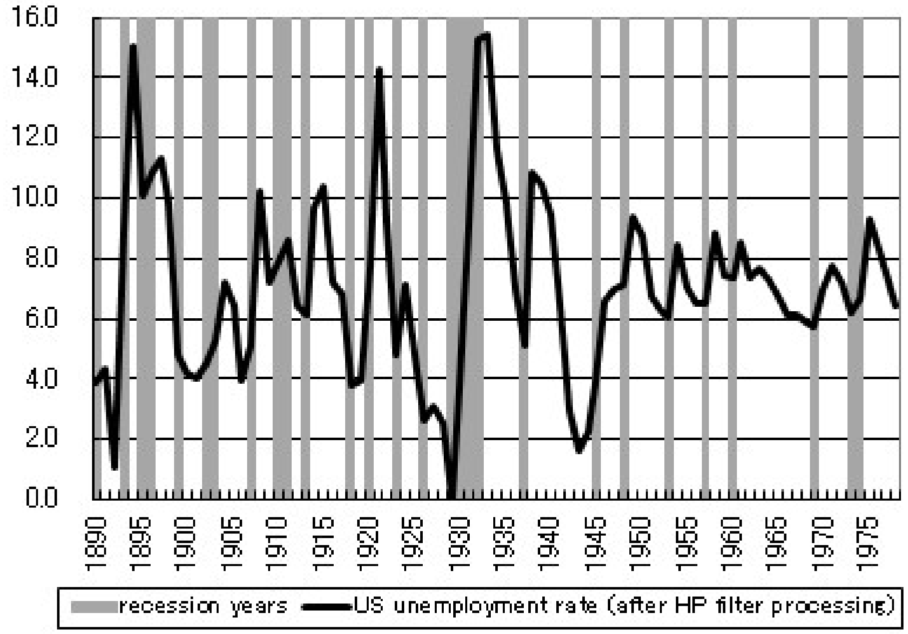

Figure 2

Transition of the growth rate of uncertainty index (gecvsq)

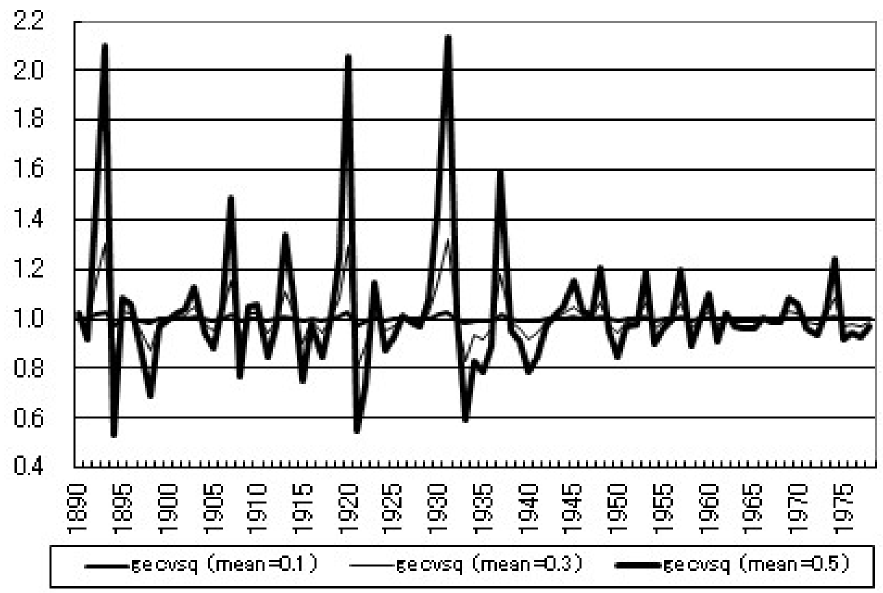

Figure 3

Derivation of the indifference curve from a two-period

expected utility function under uncertainty