Two Steady States and Two Movement Patterns under the Balanced Budget Rule- An Economy with Divisible Labor

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January 2017

Discussion Paper No.1702

GRADUATE SCHOOL OF ECONOMICS

KOBE UNIVERSITY

ROKKO, KOBE, JAPAN

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January 15, 2017

Abstract

When governments levy taxes on labor income on the basis of a balanced budget rule, two steady states in an economy exist, of which one can cause two movement patterns, namely, indeterminacy paths and a saddle path. Some of the literature deals with this issue based on an economy with indivisible labor, in which labor adjustment is made by an extensive margin. In this situation, a representative household has an infinite elasticity of substitution in regard to labor supply at different times. However, in this paper we assume an economy with divisible labor, in which labor adjustment is made by an intensive margin. We demonstrate that there indeed exist the two paths in the economy, and that there exists a critical condition dividing them. This is proved by establishing the relation between a finite elasticity of labor with regard to real wages and the share of capital in output. Consequently, we deduce the existence of an upper limitation in the share of capital in output for indeterminacy to occur. The largest possible value of that share is less than 0.5698.

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1 Introduction

In this article we demonstrate that there exist two steady states, and two different movement patterns, namely, saddle paths and indeterminate ones, in an economy with divisible labor on the assumption of labor income taxation and of a balanced budged rule.

There is a lot of literature on the issue of indeterminacy. Originally, Black (1974) and Brock (1974) point out a possibility of indeterminacy in a dynamic monetary model with a perfect foresight principle, that is, they assert that no meaningful path can be determined. In this situation, some non-fundamental factors such as expectations, which can be called sunspots or animal spirits, can determine a real economic path. Thus, an expectation can be self-fulfilling. In this sense, indeterminacy is a hot issue. Since then, many economists have located the causes of this indeterminacy in external factors in production or monopolistic competition. In line with this, we can list Benhabib and Farmer (1994). Furthermore, Kamihigashi (2002) deals with this issue in terms of externality and nonlinear discounting with respect to utility. Guo and Lansing (1998) and Wen (2001) study the problem in terms of externality and taxation and of externality and depreciation,[1] respectively. Obviously, these authors mainly seek the causes of indeterminacy in economic conditions, that is, naturally produced economic outcomes.

On the other hand, we would like to discuss indeterminacy from another viewpoint, that is, in relation to fiscal policy, as a product only of artificial determinants. In this context, fiscal policy specifically means labor income taxation and a balanced budget principle. We can list Schmitt-Grohé and Uribe (1997) in this light. Its model framework is fundamentally based on Ramsey (1928), but extends the scope to equilibria in markets. Schmitt-Grohé and Uribe (1997) examines this issue on the basis of a government balanced budget rule. And it concludes that indeterminacy can be caused within a certain range of tax rates on labor income in steady states, but not within other tax rates. Its utility function is assumed to be additively separable between consumption and leisure. Thus, its discussions regarding labor supply are based on the first category in Hansen (1985). Because we are also interested in labor income tax, we would like to analyze the mechanism of labor supply here. Hansen (1985) divides it into two categories: indivisible and divisible labor. The former means that all employees work the same number of hours while some labor is unemployed, and that the

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[1]These papers discuss a one-sector model. On the other hand, Benhabib and Farmer (1996) and Guo and Harrison (2001) discuss indeterminacy in a two-sector model.

adjustment of labor is caused by entering or leaving the labor market (the extensive margin). Full unemployment insurance is paid to individuals who are not working. In this circumstance, individuals only decide to work or not. This assumption implies that individuals face uncertainty in the labor market, that is, a possibility of unemployment. Therefore, a representative household, which has an instantaneous utility consisting of consumption and leisure, has a linear utility function with regard to labor. Of course, this conclusion is derived from the expected utility which is suitable for expressing that of a representative household. Furthermore, this linearity leads to an infinite elasticity of substitution in labor supply at different times, which shows a representative household is not averse to employment fluctuations. On the other hand, consider that all individuals are always employed. In this situation, adjustments in labor are caused by the changes in hours worked (intensive margin). Divisible labor reflects this case, and the latter category in Hansen (1985) shows this. In this situation, there is increasing marginal disutility of labor, an elasticity of labor with regard to real wages, and the elasticity of intertemporal substitution in labor supply is finite.

However, in an indivisible labor economy some individuals voluntarily quit the labor market for some reasons. In addition, unemployment insurance compensates fully. Schmitt-Grohé and Uribe (1997) takes up this case. But we think that this assumption is not necessarily realistic or comprehensive. Furthermore, the Schmitt-Grohé and Uribe framework naturally tries to deduce the supply of the labor force within its system. But any description of the labor supply is ambiguous. Since we analyze taxation on labor income, we should intensively focus on the labor supply. Motivated thus, based on the second category in Hansen (1985), we establish a model in which the household deliberately determines the supply of labor, considering an increasing disutility with regard to labor. In our paper, the role of labor supply is clearly and intensively demonstrated.

In this article, the tax rates on labor income are flexible, and, correspondingly, government expenditure is exogenously constant, on the assumption of the balanced budget rule. This system causes nonlinear dynamics.[2]

As a result, we conclude as follows: first, we demonstrate that there are two steady states in our economy, if income tax revenues equal the government expenditure given. In this process, we explicitly employ the labor supply function, which decreases when the tax rates decrease. Second,

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[2]On the other hand, there is another balanced budget rule in which tax rates remain constant while government expenditure is flexible. This system causes liner dynamics (See Guo and Harrison (2004) and Takata (2006)).

we show a saddle path and indeterminate paths in an economy and show the conditions necessary and sufficient for indeterminacy. These are shown in terms of a diagram with two elements, a reciprocal of the elasticity of labor supply with respect to real wages and the share of capital in output. Third, we prove that the ratio of capital in output to that of labor in output must be less than approximately 1.3247 for indeterminacy.

Furthermore, first, we proved that the same level of capital intensity holds in both steady states, regardless of the corresponding tax rates, which we call a *neutrality theorem*. (See the third subsection in section 3); second, we show that there are clear conditions under which maximum government revenues are decided, and these conditions are explicitly determined by parameters alone, which indicate fundamentals in an economy.

We organize this article as follows: In the next section, we introduce the framework and approach of the paper, in which the labor supply is deduced explicitly. Of course, it is affected by labor income tax rates. Then we offer a dynamic system. In what follows, in section 3, we demonstrate that there can be two steady states in that system. Subsequently, in section 4, we express our dynamic system in terms of linearity, then in section 5, show that the movement patterns are dominated by two economic variables. In section 6, initial conditions are analyzed. In section 7, we demonstrate that the two paths exist in terms of a phase diagram. Finally, section 8 is devoted to a conclusion.

2 A new model

2.1 Assumptions

An economy consists of three kinds of agents, namely households, firms and a government. Households and firms are each assumed to be represented by an agent.

A representative household possesses capital and loans it to a firm in exchange for rental fees, and additionally offers its labor to a firm. The labor population is constant and households are identical. A household faces a dynamic problem in which it determines an infinite scenario of work and consumption under budget constraints over time.

2.2 Optimization

First, we assume an instantaneous utility function of a representative household, U_t, as follows:

$$U_t = \log c_t - \frac{H_t^{1-\chi}}{1-\chi}.$$

Here, c_t means consumption and H_t working hours, respectively. A household is assumed to live forever and to maximize lifetime utility. The above utility function has the property of increasing marginal disutility with respect to H_t, contrary to models in which a utility function has linearity in labor, such as a Hansen type, including Schmitt-Grohé and Uribe (1997). Here, χ indicates a negative constant, so that $-1/\chi$ shows the elasticity of the labor supply with respect to wages. For simplicity, we postulate that the duration of capital is infinite, and as a result we omit depreciation. In this situation, a household considers labor tax rates, τ_t, real wage rates, w_t, and real rental rates of capital, u_t, as given, and solves the following dynamic problem:

$$\max_{c_t, H_t} \int_0^\infty e^{-\rho t} \left[\log c_t - \frac{H_t^{1-\chi}}{1-\chi} \right] dt,$$

subject to

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - c_t, \quad t \in \Re_+. \tag{1}$$

Here, $\Re_+$ is the set of positive reals. Additionally, ρ indicates a subjective discount rate of utility.

Second, the government observes a balanced budget rule. Since the government can observe a tax base, that is, $w_t H_t$, through the labor market, it determines a tax rate, τ_t, in order to make the labor taxes levied equal to constant government spending, G, which a balanced budged rule requires. This can be formulated as

$$G = \tau_t \ w_t \ H_t. \tag{2}$$

Third, we discuss firms. We introduce a production function F, a Cobb-Douglas type, as $F(K_t, H_t) = K_t^\alpha H_t^{1-\alpha}$, where α is a constant satisfying $0 < \alpha < 1$ and implies the share of capital in output. We assume that the markets for labor, capital, and output are fully competitive, and that a firm aims to maximize its profit. As a result, the following relationships hold:

$$w_t = \frac{\partial F}{\partial H} = F_H(K_t, H_t) = (1-\alpha) \left(\frac{K_t}{H_t} \right)^\alpha, \quad \text{and}$$

$$u_t = \frac{\partial F}{\partial K} = F_K(K_t, H_t) = \alpha \left(\frac{K_t}{H_t} \right)^{\alpha - 1}.$$

Here, output price is assumed to be 1.

At this stage, we define a Hamiltonian as

$$R = e^{-\rho t} \left[\log c_t - \frac{H_t^{1-\chi}}{1 - \chi} \right] + \mu_t [u_t K_t + (1 - \tau_t) w_t H_t - c_t].$$

We obtain the following conditions necessary for optimization:

$$\frac{\partial R}{\partial c_t} = e^{-\rho t} \frac{1}{c_t} - \mu_t = 0,$$

which can be transformed as

$$\mu_t = \frac{e^{-\rho t}}{c_t}. \tag{3}$$

On the other hand,

$$\frac{\partial R}{\partial H_t} = e^{-\rho t} \left[-H_t^{-\chi} \right] + \mu_t (1 - \tau_t) w_t = 0$$

holds. Contrary to Schmitt-Grohé and Uribe (1997), our present maximization specifically expresses the supply of labor. In short, the above equality, together with (??), enables us to know:

$$H_t = \left[\frac{(1 - \tau_t)\ w_t}{c_t} \right]^{-\frac{1}{\chi}}. \tag{4}$$

Based on the above relationship, we can easily confirm that the elasticity of this labor supply with respect to wages, e, equals $-1/\chi$, and this results in e being independent from α. This shows that H_t increases when w_t does.

Furthermore, the following holds concerning the adjoint variable μ_t:

$$\frac{d\mu_t}{dt} = -\frac{\partial R}{\partial K_t} = -\mu_t u_t,$$

which yields $\frac{\dot{\mu}_t}{\mu_t} = -u_t$. Based on the above relationship and (??), we obtain an Euler equation: $\frac{\dot{c}_t}{c_t} = u_t - \rho$. Here, we should notice that the Euler equation does not include tax rate τ_t. Moreover, for the household to project its budget constraints over time, the transversality condition needs to be

satisfied,[3] which can be expressed as $\lim_{t\to\infty} \mu_t k_t = 0$. This can be further transformed as $\lim_{t\to\infty} \frac{k_t}{c_t} e^{-\rho\, t} = 0$.

Here, we define the capital intensity of labor as $k_t = K_t/H_t$.

Of course, the transversality condition means that the present value of the marginal utility of consumption at infinity in terms of capital stock, which can be transformed into consumption through perfect substitution, must be null.[4] In addition, since R is concave with respect to c_t, H_t, μ_t and K_t, the conditions sufficient for the objective function are satisfied. In short, solutions satisfying necessary conditions are solutions for optimization.

2.3 Dynamics

Now, we need to derive the dynamic system of our economy.

First, in light of $k_t = K_t/H_t$ we obtain the following, from (??):

$$\dot{k}_t = u_t \frac{K_t}{H_t} + (1 - \tau_t) w_t - \frac{c_t}{H_t} - k_t \frac{\dot{H}_t}{H_t}. \tag{5}$$

Second, based on (??), we can express the rate of change in the labor supply as

$$\frac{\dot{H}_t}{H_t} = \frac{1}{\chi} \frac{\dfrac{\dot{c}_t}{c_t}\dfrac{1 - \tau_t}{\tau_t} + \dfrac{\dot{\tau}_t}{\tau_t} - \dfrac{1 - \tau_t}{\tau_t}\dfrac{\dot{w}_t}{w_t}}{\dfrac{1 - \tau_t}{\tau_t}}. \tag{6}$$

Here, let us consider government behavior, which is clearly described by (??), and which can be transformed as

$$\frac{\dot{\tau}_t}{\tau_t} = -\frac{\dot{w}_t}{w_t} - \frac{\dot{H}_t}{H_t}. \tag{7}$$

Substituting (??) into (??), rearranging the result and postulating that $(1 - \chi)\,\tau_t + \chi \neq 0$, we eventually obtain the rate of change in the labor supply as

$$\frac{\dot{H}_t}{H_t} = \frac{1}{(1 - \chi)\,\tau_t + \chi} \left(\frac{\dot{c}_t}{c_t}\,(1 - \tau_t) - \frac{\dot{w}_t}{w_t} \right). \tag{8}$$

---

[3]We assume that a household eventually dies at infinity, and that capital stock can be transformed into consumption with perfect substitution.

[4]See Blanchard and Fischer (1989).

Now, let us begin to calculate $\dot{k}_t$, which is described in (**??**). Then we can obtain a dynamic equation with regard to k_t as

$$
\begin{aligned}
\frac{\dot{k}_t}{k_t} =& \frac{(\alpha + (1-\alpha)(1-\tau_t))((1-\chi)\,\tau_t + \chi) - \alpha\,(1-\tau_t)}{(1-\chi)\,\tau_t + \chi\, - \alpha}\, k_t^{-(1-\alpha)} \\
& - \frac{c_t^{1-\frac{1}{\chi}}\,(1-\tau_t)^{\frac{1}{\chi}}(1-\alpha)^{\frac{1}{\chi}}((1-\chi)\,\tau_t + \chi)}{(1-\chi)\,\tau_t + \chi\, - \alpha}\, k_t^{\frac{\alpha}{\chi}-1} \\
& + \frac{\rho\,(1-\tau_t)}{(1-\chi)\tau_t + \chi - \alpha}.
\end{aligned}
\tag{9}
$$

Additionally, we can confirm $(1-\chi)\tau_t + \chi - \alpha \neq 0$.

Next, we explore a dynamic equation with regard to τ_t, which plays a critical role in our system, as will be revealed later. Based on (**??**) and (**??**), we obtain the following:

$$
\begin{aligned}
\frac{\dot{\tau}_t}{\tau_t} =& -\frac{\dot{w}_t}{w_t} - \frac{1}{(1-\chi)\,\tau_t + \chi}\,\left(\frac{\dot{c}_t}{c_t}\,(1-\tau_t) - \frac{\dot{w}_t}{w_t}\right) \\
=& \frac{1-\tau_t}{(1-\chi)\,\tau_t\, + \chi}\,\left[\alpha\,(1-\chi)\,\frac{\dot{k}_t}{k_t} - \frac{\dot{c}_t}{c_t}\right].
\end{aligned}
\tag{10}
$$

In light of (**??**) and the Euler equation, (**??**) can be transformed into (**??**):

$$
\begin{aligned}
\frac{\dot{\tau}_t}{\tau_t} =& \frac{1-\tau_t}{(1-\chi)\tau_t + \chi}\,[\alpha\frac{(\chi + (1-\chi)\tau_t)((\alpha-\chi) - (1-\chi)(1-\alpha)\tau_t)}{(1-\chi)\tau_t + \chi - \alpha}k_t^{\alpha-1} \\
& - \frac{\alpha(1-\chi)c_t^{1-\frac{1}{\chi}}(1-\tau_t)^{\frac{1}{\chi}}(1-\alpha)^{\frac{1}{\chi}}((1-\chi)\tau_t + \chi)}{(1-\chi)\tau_t + \chi - \alpha}k_t^{\frac{\alpha}{\chi}-1} \\
& + \frac{\rho(1-\alpha)((1-\chi)\tau_t + \chi)}{(1-\chi)\tau_t + \chi - \alpha}].
\end{aligned}
\tag{11}
$$

As a consequence, we can establish our dynamic system consisting of (**??**), the Euler equation and (**??**).

At this stage, we introduce new variables: $\log k_t = \lambda_t$, $\log c_t = \delta_t$, and $\log \tau_t = \eta_t$. In this situation, the system consisting of (**??**), the Euler equation and (**??**) can be expressed as follows:

$$
\dot{\delta}_t = \alpha e^{-(1-\alpha)\lambda_t} - \rho.
\tag{12}
$$

8

From this we can extrapolate the following:

$$\dot{\lambda}_t = \frac{(\alpha + (1-\alpha)(1-e^{\eta_t}))((1-\chi)e^{\eta_t} + \chi) - \alpha(1-e^{\eta_t})}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\alpha-1)\lambda_t}$$
$$- \frac{e^{\frac{\chi-1}{\chi}\delta_t}(1-e^{\eta_t})^{\frac{1}{\chi}}(1-\alpha)^{\frac{1}{\chi}}((1-\chi)e^{\eta_t} + \chi)}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\frac{\alpha}{\chi}-1)\lambda_t}$$
$$+ \frac{\rho(1-e^{\eta_t})}{(1-\chi)e^{\eta_t} + \chi - \alpha}, \tag{13}$$

and

$$\dot{\eta}_t = \frac{1-e^{\eta_t}}{(1-\chi)e^{\eta_t} + \chi}\left[\frac{\alpha(\chi + (1-\chi)e^{\eta_t})((\alpha-\chi) - (1-\chi)(1-\alpha)e^{\eta_t})}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\alpha-1)\lambda_t}\right.$$
$$- \frac{\alpha(1-\alpha)^{\frac{1}{\chi}}(1-\chi)e^{\frac{\chi-1}{\chi}\delta_t}(1-e^{\eta_t})^{\frac{1}{\chi}}((1-\chi)e^{\eta_t} + \chi)}{(1-\chi)e^{\eta_t} + \chi - \alpha} e^{(\frac{\alpha}{\chi}-1)\lambda_t}$$
$$+ \left.\frac{\rho(1-\alpha)((1-\chi)e^{\eta_t} + \chi)}{(1-\chi)e^{\eta_t} + \chi - \alpha}\right]. \tag{14}$$

This system, (??) through (??), consists of three variables, δ_t, λ_t and η_t and the three corresponding differential equations. Notice that the system does not include government spending G.

3 Existence of two equilibria

3.1 Two equilibria

Assume that there is a steady state in our system, denoted by δ^*, λ^* and η^*. First, by making $\dot{\delta}_t = 0$ in (??), we obtain

$$k^* = e^{\lambda^*} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}, \tag{15}$$

which shows that the capital intensity of labor in steady states is unique and determined by two parameters alone, α and ρ, independently of labor tax rates. Second, in a similar fashion, as $\dot{\delta}_t = 0$, by making $\dot{\lambda}_t = 0$ in (??) and $\dot{\eta}_t = 0$ in (??), we obtain the following: When $\dot{\lambda}_t = 0$,

$$[(\alpha + (1-\alpha)x)((\chi-1)x + 1) - \alpha x]\left(\frac{\rho}{\alpha}\right)$$
$$- x^{\frac{1}{\chi}}(1-\alpha)^{\frac{1}{\chi}}((\chi-1)x + 1)\,q\,e^{\frac{\alpha-\chi}{\chi}\lambda^*} + \rho x = 0, \tag{16}$$

and when $\dot{\eta}_t = 0$,

$$\alpha \ (\chi + (1 - \chi)(1 - x))((\alpha - \chi) - (1 - \chi)(1 - \alpha)(1 - x)) \left(\frac{\rho}{\alpha}\right)$$
$$- \alpha \ (1 - \alpha)^{\frac{1}{\chi}}(1 - \chi) \ q \ x^{\frac{1}{\chi}}((\chi - 1) \ x + 1) \ e^{\frac{\alpha - \chi}{\chi}\lambda^*}$$
$$+ \rho \ (1 - \alpha)((\chi - 1)x + 1) = 0. \tag{17}$$

Additionally, to have a clear perspective, we transformed variables as $1 - e^{\eta^*} = x$, and $e^{\frac{\chi-1}{\chi}\delta^*} = q$. Here, because $0 < e^{\eta^*} = 1 - x = \tau^* < 1$, $0 < x < 1$. In this situation, we consider x and q as variables.

We consider the above system, (**??**) and (**??**), as a simultaneous equation system concerning x and q. However, we can easily confirm that (**??**) $\times \alpha(1 - \chi) = $ (**??**). This implies that (**??**) and (**??**) are not independent of each other. Therefore, we seek another relationship including x and q. We need to focus on the balanced budget in a steady state, which is described as

$$G = \tau^* \ w^* \ H^* = \tau^* \ w^* \left[\frac{(1 - \tau^*) \ w^*}{c^*}\right]^{-\frac{1}{\chi}}. \tag{18}$$

Notice that (**??**) is originally deduced from the balanced budget rule, and therefore the dynamic equation concerning τ is not independent (See (**??**)).

Rearranging (**??**), we obtain the following:

$$q = \left[\frac{G^\chi}{(w^*)^{\chi-1}} \ \frac{x}{(1 - x)^\chi}\right]^{\frac{\chi-1}{\chi}}. \tag{19}$$

Substituting (**??**) into (**??**), we obtain an equation only in terms of x as

$$[(\alpha + (1 - \alpha)x)((\chi - 1) \ x + 1) - \alpha \ x]\left(\frac{\rho}{\alpha}\right) + \rho \ x$$
$$- (1 - \alpha)^{\frac{1}{\chi}} \ e^{\frac{\alpha - \chi}{\chi}\lambda^*}((\chi - 1) \ x + 1) \tag{20}$$
$$\times x^{\frac{1}{\chi}}\left[\frac{G^\chi}{(w^*)^{\chi-1}}\right]^{\frac{\chi-1}{\chi}} \left[\frac{x}{(1 - x)^\chi}\right]^{\frac{\chi-1}{\chi}} = 0,$$

which can further be transformed as

$$[(\alpha + (1 - \alpha)x)((\chi - 1) \ x + 1) - \alpha \ x]\left(\frac{\rho}{\alpha}\right) + \rho \ x$$
$$- G^{\chi-1} \ (1 - \alpha)^{2-\chi} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(2-\chi)-1}{1-\alpha}} ((\chi - 1)x + 1)(1 - x)^{1-\chi} \ x = 0. \tag{21}$$

When we denote the left side in the above equation as $h(x)$, that can be expressed as

$$\begin{cases} h(x) = (\chi - 1)\left(x - \dfrac{1}{1 - \chi}\right)(1 - \alpha)\varphi(x), \\ \varphi(x) = x\left[1 - \Omega(1 - x)^{1-\chi}\right] + \dfrac{\alpha}{1-\alpha}, \\ \Omega = (1 - \alpha)^{1-\chi}G^{\chi - 1}\left(\dfrac{\alpha}{\rho}\right)^{\frac{\alpha(1-\chi)}{1-\alpha}} = \left(\dfrac{w^*}{G}\right)^{1-\chi} > 0. \end{cases} \tag{22}$$

The solution in the above equation, $h(x) = 0$, shows the steady state which we are investigating. Of course, $0 < x < 1$ must be satisfied. From what (**??**) implies, since x which satisfies $h(x) = 0$ is a possible solution, $x = 1/(1 - \chi)$ seems to be a solution on appearance. However, if $x = 1/(1 - \chi)$ holds, $(1 - \chi)\tau^* + \chi = 0$ is satisfied, which we have rejected (See (**??**)).

At this stage, it is obvious that the steady state is a solution satisfying $\varphi(x) = 0$. Therefore, we need to examine how $\varphi(x)$ behaves in the range of $0 < x < 1$. We know that the curve has a minimum point at $0 < x^* < 1$. So, if the curve at the minimum point is negative, our system can have two steady states.[5] Based on Appendix, we offer an example in Figure 1.

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[5]For details, see Appendix . Additionally, there is a possibility that the system has a double steady state. We deal with this issue in the second subsection of this section.
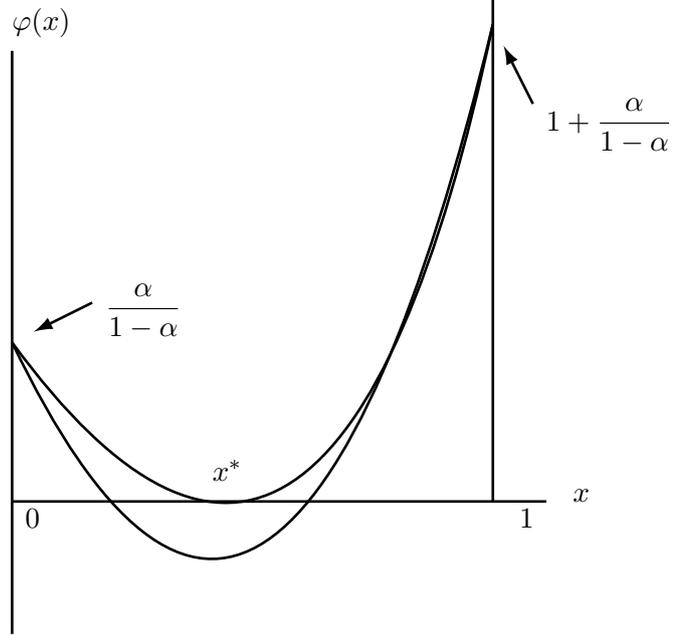
Figure 1: Two **x**s can exist; if $\varphi(\mathbf{x}^*) < \mathbf{0}$, two steady states exist; in the case of $\varphi(\mathbf{x}^*) = \mathbf{0}$, a double steady state exists.

As we have discussed so far, assuming an increasing marginal disutility with regard to labor, we can offer some conditions for two steady states as a theorem:

Let x^* be a solution to the following equation, as is demonstrated in Appendix 1:

$$1 - (2 - \chi)x^* = \frac{1}{\Omega(1 - x^*)^{-\chi}}. \qquad (23)$$

In this situation, we can show the existence of two steady states.

Theorem 1 (The existence of two steady states) *There exist two steady states if the following conditions are satisfied:*

$$\begin{cases} (a) & \Omega > 1, \\ (b) & x^*[1 - \Omega(1 - x^*)^{1-\chi}] + \frac{\alpha}{1-\alpha} \leqslant 0, \\ (c) & K^\alpha H^{1-\alpha} = c + G. \end{cases}$$

Additionally, condition (c) reflects Walras' law and shows the values in the steady states.

When a unique G is given, in the case of inequality in (b), there can correspondingly be two steady states, while in the case of equality in (b), a double root in relation to x^* appears. This implies that two similar steady states can coexist. In the second subsection in this section, we show that the system satisfying *Theorem 1* is not empty. *Theoerm 1* is established based on a particular x^* satisfying (**??**). However, we can demonstrate the same relationship as in *Theorem 1* for an arbitrary situation in which $0 < x < 1$.

Corollary 1 (The existence of two steady states) *There exist two steady states if the following conditions are satisfied:*

$$\begin{cases} (d) & \Omega = \dfrac{x + \frac{\alpha}{1-\alpha}}{x(1-x)^{1-\chi}}, \\ (e) & K^\alpha H^{1-\alpha} = c + G. \end{cases}$$

Since the right hand side of (d) in *Corollary 1* is larger than unity for any value of $0 < x < 1$, the condition $\Omega > 1$ in (a) in *Theorem 1* is necessarily satisfied in *Corollary 1*. In this situation, we need to elucidate (d) in more detail, from an economic viewpoint. Transforming (d), we obtain

$$G = w^*(1-x) \left(\frac{x}{x + \frac{\alpha}{1-\alpha}} \right)^{\frac{1}{1-\chi}}. \tag{24}$$

The above equation determines x, i.e. τ^*. The right side in (**??**) shows labor tax revenue in terms of x alone[6] and is independent of G. Now, let us focus on the shape of this labor tax revenue function $\zeta(\tau^*) = w^* \ \tau^* \ h(\tau^*)$, where $h(\tau^*)$ is deduced as

$$h(\tau^*) = \left(\frac{1 - \tau^*}{1 - \tau^* + \frac{\alpha}{1-\alpha}} \right)^{\frac{1}{1-\chi}},$$

which indicates the labor supply in terms of τ^* alone. This is significant for the development of later discussions. We can consider it as the labor supply function based on τ^* alone. Note that $x = 1 - \tau^*$. At this stage, let us explore $h(\tau^*)$ more intensively. It is apparent that $h(\tau^*)$ can shift up and down depending on parameters α and χ. Obviously, $\dfrac{\alpha}{1-\alpha}$ shows the ratio of capital income in output to that of labor income in output. And this ratio

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[6]Note that w^* is equal to the marginal product of labor in our two steady states and is constant.

implies that the income from working is comparatively more advantageous than that from renting capital, when α is small. $h(\tau^*)$ reflects this kind of incentive in relation to labor supply. Of course, if α increases (decreases), the labor supply shifts downward (upward). On the other hand, as χ decreases, $h(\tau^*)$ shifts upward, and vice versa. This implies that the labor supply is in inverse proportion to the elasticity of labor supply in relation to wages. Specifically, the larger the elasticity is, the less $h(\tau^*)$. Moreover, note that $h(\tau^*)$ is clearly less than 1.

Based on the definition of $\zeta(\tau^*)$, we obtain the following:

$$\frac{d\zeta}{d\tau^*} = w^* h(\tau^*) \left(1 + \frac{\tau^*}{h(\tau*)} \frac{dh(\tau^*)}{d\tau^*} \right) = w^* h(\tau^*)(1 + \epsilon(\tau*)),$$

where we define $\epsilon(\tau*) = \dfrac{\tau*}{h(\tau*)} \dfrac{dh(\tau*)}{d\tau^*}$. $\epsilon(\tau^*)$ indicates the elasticity of labor supply in relation to labor income tax rates and has the following properties: $\epsilon'(\tau*) = \dfrac{-\frac{\alpha}{1-\alpha}\tau^*}{(1-\tau^*)(1-\chi)(1-\tau^* + \frac{\alpha}{1-\alpha})} < 0;\ \epsilon(0) = 0; \epsilon(1) \to -\infty.$ This implies that as τ^* increases, the labor supply decreases drastically. In this situation, in ranges of comparatively small τ^*, since the increment ratio in τ^* is larger than the rate of decrease in $w^* H$, i.e., the tax base, tax revenue results in an increase. On the other hand, in ranges of comparatively larger τ^*, since the increment rate in τ^* is smaller than the rate of decrease in the tax base, the tax revenue results in a decrease. Therefore, when $0 < \tau^* \leqslant \tau^{**}, \epsilon + 1 \geqslant 0$, while in ranges of $\tau^{**} < \tau^* < 1, \epsilon + 1 < 0$. This indicates that when $0 < \tau^* \leqslant \tau^{**}, \dfrac{d\zeta}{d\tau^*} \geqslant 0$, and when $\tau^{**} < \tau^* < 1, \dfrac{d\zeta}{d\tau^*} < 0$.

Additionally, τ^{**} shows the labor income tax rate corresponding to maximum tax revenue. Furthermore, $\zeta(\tau^*)$ has the following properties: $\zeta(0) = 0; \zeta(1) = 0, \zeta > 0,$ and $\frac{d^2\zeta}{d(\tau^*)^2} < 0,[7]$ for $0 < \tau^* < 1$. And ζ is a continuous function with respect to τ^*.

From what we have described so far, it is obvious that the shape of ζ is like that of a bell curve.[8] This bell curve is independent of G. Therefore, we can say that there are two τ^*s when G is given. In other words, there

---

[7] A proof is omitted.

[8] Schmitt-Grohé and Uribe deduce this kind of bell curve avoiding H itself, instead considering K, i.e. savings, which is a function with τ^* as an argument. In an indivisible labor type model, such as Schmitt-Grohé and Uribe (1997) or Anagnostopoulos and Giannitsarou (2013), an economic system can be described in terms of the capital-labor ratio. Once the capital intensity of labor is determined, what follows is the determination of capital, thus capital is in fact predetermined. That is, labor is automatically determined by capital intensity. In this sense, the labor supply is arbitrary, and in this situation, the

are two alternatives in relation to τ^* which correspond to the government spending financed by this tax revenue. *Theorem 1* shows that there is a bell curve in our system, which causes nonlinear dynamics. In this context, G must naturally be less than or equal to the maximum tax revenue, which is determined by fundamental parameters alone. (See the second subsection.) This is what *Corollary 1* shows. In other words, G is restricted by the fundamentals in the system. Since the tax revenue function depends on the labor supply, and the labor supply function on χ and α, we know that when χ is larger, the corresponding tax revenue function is located under the functions that correspond to smaller χ, for $0 < \tau^* < 1$, if α is kept unchanged. Specifically, when χ is approximately null, the tax revenue function is the smallest.

3.2   Maximum G

At this stage, let us focus on the relationship between G and our two xs. We can easily know from the above analyses that if G increases, $\varphi(x)$ tends to go upwards, which is caused by a decrement of Ω. And this leads to a double root or no root with regard to x. This implies an upper limitation with regard to G in our context. Of course, maximum G corresponds to maximum labor income tax revenue. We can find the maximum G as follows: First, since $\varphi'(x^*) = 0$ must be satisfied, we obtain

$$\Omega = \frac{(1 - x^*)^\chi}{1 - (2 - \chi)x^*}. \tag{25}$$

Second, under the condition of the above Ω, $\varphi(x^*) = 0$ must be also satisfied. Therefore, the following relationship holds:

$$x^* \left[1 - \frac{(1 - x^*)^\chi (1 - x^*)^{1-\chi}}{1 - (2 - \chi)x^*} \right] + \frac{\alpha}{1 - \alpha} = 0,$$

which can be transformed as $(\chi - 1)(x^*)^2 - \dfrac{\alpha(2 - \chi)}{1 - \alpha}x^* + \dfrac{\alpha}{1 - \alpha} = 0$.

Solving the above quadratic equation, we eventually obtain the solution required as

$$x^* = \frac{\alpha(2 - \chi) - \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1 - \alpha)(\chi - 1)} > 0. \tag{26}$$

---

agent which determines labor in an equilibrium is the firm. (See Schmitt-Grohé and Uribe (1997), page 981). Note that in an indivisible labor model, the elasticity of labor supply in relation to wages is infinite.

Substituting (**??**) into (**??**), Ω is determined by parameters such as α and χ alone, and this shows the maximum G. Thus,

$$G = (1 - \alpha) \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \left[\frac{1 - (2 - \chi)x^*}{(1 - x^*)^\chi}\right]^{\frac{1}{1-\chi}}$$

is derived. Moreover, in this context, we can demonstrate that the values in this equilibrium of c, τ and H are explicitly determined by fundamental parameters alone, such as α, ρ and χ, as G is. We offer only the results of this determination as follows[9].

$$\begin{cases} \tau^* = \dfrac{\chi(2 - \alpha) - 2 + \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1 - \alpha)(\chi - 1)}, \\[2ex] G = (1 - \alpha) \left(\dfrac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \left[\dfrac{(\tau^*)^\chi}{1 - (2 - \chi)(1 - \tau^*)}\right]^{-\frac{1}{1-\chi}}, \\[2ex] c^* = (1 - \alpha) \left(\dfrac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} (1 - \tau^*) \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)}\right]^{-\frac{\chi}{1-\chi}}, \\[2ex] H^* = \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)}\right]^{-\frac{1}{1-\chi}}. \end{cases}$$

Of course, k^* can be derived in the same way. Thus, we can easily describe K^*, u^* and Y^* by parameters alone as well. In the same vein, we can establish the following theorem:

3.3 Neutrality

Capital intensity was shown as

$$k^* = e^{\lambda^*} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}.$$

Theorem 2 *Capital intensity in steady states is independent of tax rates. All k^*s are equal regardless of G.*

---

[9]After the seventh section, this tax rate corresponding to the maximum tax revenue can be denoted by τ^{**}.

4 Linearization

Based on what we have analyzed so far, we can express our dynamic system as follows:

$$\begin{cases} \dot{\delta}_t = f^1(\lambda_t), \\ \dot{\lambda}_t = f^2(\delta_t, \lambda_t, \eta_t), \\ \dot{\eta}_t = f^3(\delta_t, \lambda_t, \eta_t). \end{cases} \tag{27}$$

Linearizing (??) around the steady state $(\delta^*, \lambda^*, \eta^*)$, we obtain the following simpler expressions:

$$\begin{cases} \dot{\mathbf{z}} = A\mathbf{z}, \\ \mathbf{z} = [z_1, \quad z_2, \quad z_3]' = [\delta_t - \delta^*, \quad \lambda_t - \lambda^*, \quad \eta_t - \eta^*]'. \end{cases} \tag{28}$$

Here, the notation $'$ shows the transposition of a vector. A is defined as

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Moreover, we define the elements in A as follows:

$$a_{12} = \frac{\partial f^1}{\partial \lambda}, \quad a_{21} = \frac{\partial f^2}{\partial \delta}, \quad a_{22} = \frac{\partial f^2}{\partial \lambda}, \quad a_{23} = \frac{\partial f^2}{\partial \eta}$$

$$a_{31} = \frac{\partial f^3}{\partial \delta}, \quad a_{32} = \frac{\partial f^3}{\partial \lambda}, \quad a_{33} = \frac{\partial f^2}{\partial \eta}.$$

In addition, let us return to (??), which shows that:

$$\dot{\eta}_t = \frac{1 - e^{\eta_t}}{(1 - \chi)e^{\eta_t} + \chi} \left[\alpha(1 - \chi)f^2(\delta_t, \lambda_t, \eta_t) - f^1(\lambda_t) \right].$$

From the above equation, we obtain the following:

$$\begin{cases} \dfrac{\partial f^3}{\partial \delta_t} = \dfrac{\alpha(1 - \chi)x}{(\chi - 1)x + 1} \dfrac{\partial f^2}{\partial \delta_t}, \\ \dfrac{\partial f^3}{\partial \lambda_t} = \dfrac{x}{(\chi - 1)x + 1} \left[\alpha(1 - \chi)\dfrac{\partial f^2}{\partial \lambda_t} - \dfrac{\partial f^1}{\partial \lambda_t} \right], \\ \dfrac{\partial f^3}{\partial \eta_t} = \dfrac{\alpha(1 - \chi)x}{(\chi - 1)x + 1} \dfrac{\partial f^2}{\partial \eta_t}. \end{cases} \tag{29}$$

Here, we can express (??) in terms of a_{ij}, Δ and Δ^*, as $a_{31} = \Delta\, a_{21}$, $a_{32} = \Delta\, a_{22} - \Delta^*\, a_{12}$, and $a_{33} = \Delta\, a_{23}$. Additionally, Δ and $\Delta*$ are respectively

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defined as $\Delta = \dfrac{\alpha(1-\chi)x}{(\chi-1)x+1}$ and $\Delta^* = \dfrac{x}{(\chi-1)x+1}$. Of course, in this context, x is a constant in terms of η^*. Note that $x = 1 - e^{\eta^*}$.

Thus, we can express A as

$$
A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ \Delta\, a_{21} & \Delta\, a_{22} - \Delta^*\, a_{12} & \Delta\, a_{23} \end{pmatrix}. \tag{30}
$$

This matrix controls the movement patterns in our fundamental system.

5 Movement patterns

At this stage, we need to detect both eigenvalues and corresponding eigenvectors in A, because we want to investigate how our economy moves under a fundamental economic condition.

5.1 Eigenvalues and eigenvectors

First, we seek eigenvalues of the Jacobian matrix with 3×3 elements in (??). We denote them as θ. After some calculations, we obtain a characteristic polynomial as

$$
\phi(\theta) = \left| \theta I - A \right| = -\theta[\theta^2 - (a_{22} + \Delta\, a_{23})\theta + a_{12}(\Delta^* a_{23} - a_{21})].
$$

Based on the above polynomial, we obtain three eigenvalues as follows:

$$
\begin{cases}
\theta_1 = 0, \\
\theta_2 = \dfrac{1}{2}(a_{22} + \Delta\, a_{23} - \sqrt{(a_{22} + \Delta\, a_{23})^2 + 4a_{12}(a_{21} - \Delta^*\, a_{23})}), \\
\theta_3 = \dfrac{1}{2}(a_{22} + \Delta\, a_{23} + \sqrt{(a_{22} + \Delta\, a_{23})^2 + 4a_{12}(a_{21} - \Delta^*\, a_{23})}).
\end{cases}
$$

Second, we seek corresponding eigenvectors.

(1) As for $\theta_1 = 0$,

the eigenvector can be described as

$$
[h_1^1,\ h_1^2,\ h_1^3]' = \left[-\frac{a_{23}}{a_{21}},\ 0,\ 1 \right]'.
$$

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(2) As for θ_i, the following holds[10] :

$$
\begin{pmatrix}
-\theta_i & a_{22} & 0 \\
a_{21} & a_{22} - \theta_i & a_{23} \\
\Delta\, a_{21} & \Delta\, a_{22} - \Delta^*\, a_{12} & \Delta\, a_{23} - \theta_i
\end{pmatrix}
\begin{pmatrix}
h_i^1 \\
h_i^2 \\
h_i^3
\end{pmatrix} = 0.
$$

Additionally, $i = 2, 3$. Manipulating the above formula, we eventually obtain the following eigenvectors required:

$$
[h_i^1,\ h_i^2,\ h_i^3]' = \left[\frac{a_{12}}{\Delta\theta_i - a_{12}\Delta^*}\ ,\quad \frac{\theta_i}{\Delta\theta_i - a_{12}\Delta^*}\ ,\quad 1 \right]' ,
$$

where we assume $\theta_2 \neq \theta_3$.

We need to define the following matrix, P, which consists of the eigenvectors derived in the calculations above:

$$
P = \begin{pmatrix}
-\dfrac{a_{23}}{a_{21}} & \dfrac{a_{12}}{\Delta\theta_2 - a_{12}\Delta^*} & \dfrac{a_{12}}{\Delta\theta_3 - a_{12}\Delta^*} \\[2ex]
0 & \dfrac{\theta_2}{\Delta\theta_2 - a_{12}\Delta^*} & \dfrac{\theta_3}{\Delta\theta_3 - a_{12}\Delta^*} \\[2ex]
1 & 1 & 1
\end{pmatrix}. \tag{31}
$$

Now, we can derive the inverse of P as follows:

$$
P^{-1} = \begin{pmatrix}
\dfrac{a_{21}\,\Delta^*}{R} & \dfrac{-a_{21}\,\Delta}{R} & \dfrac{a_{21}}{R} \\[2ex]
\dfrac{a_{21}\,\theta_3(-\Delta\,\theta_2 + a_{12}\,\Delta^*)}{a_{12}(\theta_2 - \theta_3)R} & \dfrac{(\Delta\,\theta_2 - a_{12}\,\Delta^*)(\Delta\,a_{23}\theta_3 + a_{12}R)}{a_{12}(\,\theta_2 - \theta_3)R} & \dfrac{a_{23}\,\theta_3(-\Delta\,\theta_2 + a_{12}\,\Delta^*)}{a_{12}(\theta_2 - \theta_3)R} \\[2ex]
\dfrac{a_{21}\,\theta_2(\Delta\,\theta_3 - a_{12}\,\Delta^*)}{a_{12}(\theta_2 - \theta_3)R} & -\dfrac{(\Delta\,\theta_3 - a_{12}\,\Delta^*)(\Delta\,a_{23}\theta_2 + a_{12}R)}{a_{12}(\,\theta_2 - \theta_3)R} & \dfrac{a_{23}\,\theta_2(\Delta\,\theta_3 - a_{12}\,\Delta^*)}{a_{12}(\theta_2 - \theta_3)R}
\end{pmatrix},
$$

where $R = a_{21} - a_{23}\,\Delta^*$.[11]

5.2   Transformation of the original system

Based on the arguments so far, we introduce a Jordan canonical form of A. Multiplying both sides in (??) by P^{-1}, which is defined from (??) , we obtain the following relationship, $P^{-1}\dot{\mathbf{z}} = P^{-1}A\,\mathbf{z}$, and therefore $\mathbf{y} = P^{-1}\mathbf{z}$.[12] Thus, the system with regard to $\mathbf{y}$ is established as

$$
\dot{\mathbf{y}} = P^{-1}AP\mathbf{y} = J(A)\mathbf{y}. \tag{32}
$$

[10]We note the elements in the eigenvectors as h_i^j, below. Here, i indicates the number identifying eigenvalues, and j the number showing places within the corresponding eigenvector in A, respectively.

[11]Note that P^{-1} exists and is regular, because we assume that $a_{21} - a_{23}\Delta^* \neq 0$

[12]$\mathbf{y}$ indicates a vector with three rows and one column, and is expressed as $\mathbf{y} = [y_1, y_2, y_3]'$.

Since $J(A)$ is obviously a Jordan canonical form of A, the movements with regard to $\mathbf{z}$ in the original system are in the same vein as those with regard to $\mathbf{y}$. So, we investigate system (??) below. Under this assumption, when we denote (??) in more detail, the following hold:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{33}$$

Solving (??), we have solutions as follows:

$$\begin{cases} y_1 = b \quad (\text{constant}), \\ y_2 = y_2(0)e^{\theta_2 t}, \\ y_3 = y_3(0)e^{\theta_3 t}. \end{cases} \tag{34}$$

Here, $y_2(0)$ and $y_3(0)$ show initial conditions concerning y_2 and y_3, respectively.

Before we analyze the whole system described in (??), first of all, we can prove $y_1(t) = 0$. In this situation, we can consider that the system is autonomously controlled by only two variables, y_2 and y_3. This means that we can concretely focus on the following system alone:

$$\begin{pmatrix} \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_3 \end{pmatrix} \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}. \tag{35}$$

The characteristic polynomial from this system is obviously shown as

$$\theta^2 - (a_{22} + \Delta\ a_{23})\theta + a_{12}(\Delta^* a_{23} - a_{21}).$$

Now, we need to explore our system more intensively. Based on P, we can establish our system through linear transformations of $P\mathbf{y} = \mathbf{z}$, as follows:

$$P\mathbf{y} = \begin{pmatrix} -\dfrac{a_{23}}{a_{21}} & \dfrac{a_{12}}{\Delta\theta_2 - a_{12}\Delta^*} & \dfrac{a_{12}}{\Delta\theta_3 - a_{12}\Delta^*} \\ 0 & \dfrac{\theta_2}{\Delta\theta_2 - a_{12}\Delta^*} & \dfrac{\theta_3}{\Delta\theta_3 - a_{12}\Delta^*} \\ 1 & \dfrac{1}{\Delta\theta_2 - a_{12}\Delta^*} & \dfrac{1}{\Delta\theta_3 - a_{12}\Delta^*} \end{pmatrix} \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \dfrac{a_{12}}{\Delta\theta_2 - a_{12}\Delta^*}y_2 + \dfrac{a_{12}}{\Delta\theta_3 - a_{12}\Delta^*}y_3 \\ \dfrac{\theta_2}{\Delta\theta_2 - a_{12}\Delta^*}y_2 + \dfrac{\theta_3}{\Delta\theta_3 - a_{12}\Delta^*}y_3 \\ y_2 + y_3 \end{pmatrix}$$

$$= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Thus, we obtain the following relationships:

$$\begin{cases} z_1 = \dfrac{a_{12}}{\Delta\theta_2 - a_{12}\Delta^*}y_2 + \dfrac{a_{12}}{\Delta\theta_3 - a_{12}\Delta^*}y_3, \\ z_2 = \dfrac{\theta_2}{\Delta\theta_2 - a_{12}\Delta^*}y_2 + \dfrac{\theta_3}{\Delta\theta_3 - a_{12}\Delta^*}y_3, \\ z_3 = y_2 + y_3. \end{cases}$$

Moreover, the following hold:[13] $\Delta\, z_2 - \Delta^* z_1 = y_2 + y_3$, $z_3 = y_1 + y_2 + y_3$, and $y_1 = 0$. In short, this eventually means $z_3 = \Delta z_2 - \Delta^* z_1$ in terms of $\mathbf{z}$. Thus, it becomes obvious that z_3 is a linear dependence, following z_1 and z_2. From what we have analyzed so far, we can say that when y_2 and y_3 are determined, z_1 and z_2 are correspondingly determined. Thus, z_3 is also determined. In this context, it is clear that movements by y_2 and y_3 control all the movements in our system.

6   Determination of initial conditions

Our system consists of four ingredients, one state variable, K, and three adjoint variables, H, c, and τ, and we discuss the ratio of the state variable and one adjoint one, K/H. However, the conditions for indeterminacy can be expressed by (**??**) and (**??**), as verified.

7   Indeterminacy paths and a saddle path

Based on the arguments in the last section, we have established the possibility of two different movement patterns, namely indeterminacy paths and a saddle path, and the existence of two steady states in our system. In this section, we intend to demonstrate that one path toward one bigger steady state in terms of the tax rates is a saddle path, while other paths toward a smaller steady state in terms of tax rates can be indeterminate paths, on the assumption of government spending as a given. We have established that our system movements are controlled by (**??**). In terms of this system, we explore conditions under which indeterminacy occurs. In our system, the following conditions are necessary and sufficient for indeterminacy:

$$\theta_2 + \theta_3 = a_{22} + \Delta a_{23} < 0 \tag{36}$$

and

$$\theta_2\theta_3 = a_{12}\left(\Delta^* a_{23} - a_{21}\right) > 0. \tag{37}$$

Below, we show that the constants in the right side in the above relationships are respectively expressed by a function of τ^*, except for a_{12}. We obtain the following:

---

[13]The first relationship is derived from $\mathbf{y} = P^{-1}\mathbf{z}$, which was defined.

$$
\left\{
\begin{aligned}
&\Delta(\tau^*) = \frac{\alpha(1-\tau^*)(1-\chi)}{1+(1-\tau^*)(-1+\chi)}, \\
&\Delta^*(\tau^*) = \frac{1-\tau^*}{1+(1-\tau^*)(-1+\chi)}, \\
&a_{12} = (-1+\alpha)\rho, \\
&a_{21}(\tau^*) = \frac{k^{-1+\frac{\alpha}{\chi}}w^{\frac{-1+\chi}{\chi}}(1-\alpha)^{\frac{1}{\chi}}(\tau^*)^2(-1+\chi)^2}{(\alpha-\chi)\chi+\tau^*(-1+\chi)\chi} \\
&\quad +\frac{(1+\kappa)k^{-1+\frac{\alpha}{\chi}}w^{\frac{-1+\chi}{\chi}}(1-\alpha)^{\frac{1}{\chi}}(-1+\chi)\chi}{(\alpha-\chi)\chi+\tau^*(-1+\chi)\chi} \\
&\quad +\frac{k^{-1+\frac{\alpha}{\chi}}w^{\frac{-1+\chi}{\chi}}(1-\alpha)^{\frac{1}{\chi}}\tau^*(-1+\chi)(1+\kappa-2\chi-\kappa\chi)}{(\alpha-\chi)\chi+\tau^*(-1+\chi)\chi}, \\
&a_{22}(\tau^*) = \frac{\left(k^{\alpha-1}(-1+\alpha)+k^{-1+\alpha/\chi}w^{\frac{-1+\chi}{\chi}}(1-\alpha)^{\frac{1}{\chi}}(1+\kappa)\right)(-\alpha+\chi)}{-\alpha+\tau^*(1-\chi)+\chi} \\
&\quad -\frac{(\tau^*)^2(-1+\chi)\left(w^{1-1/\chi}k^{-1+\alpha/\chi}\chi^{-1}(1-\alpha)^{\frac{1}{\chi}}(\alpha-\chi)+k^{-1+\alpha}(-1+\alpha)^2\right)}{-\alpha+\tau^*(1-\chi)+\chi} \\
&\quad +\frac{k^{-1+\frac{\alpha}{\chi}}w^{\frac{-1+\chi}{\chi}}(1-\alpha)^{\frac{1}{\chi}}\tau^*(\alpha-\chi)(-1+\kappa(-1+\chi)+2\chi)\chi^{-1}}{\alpha+\tau^*(1-\chi)+\chi} \\
&\quad +\frac{k^{\alpha}w^{\frac{1}{\chi}}(-1+\alpha)\chi(1+\alpha+(-2+\alpha)\chi)}{-\alpha+\tau^*(1-\chi)+\chi}, \\
&a_{23}(\tau^*) = [(\tau^*\tfrac{1}{k}(w^{-1/\chi}(1-\tau^*)^{-\frac{1}{(1-\chi)\chi}}(1+\kappa-\tau^*)^{\frac{1}{1-\chi}} \\
&\quad \times(-k^{\alpha/\chi}w(1-\alpha)^{\frac{1}{\chi}}(1-\tau^*)^{\frac{1}{(1-\chi)\chi}}(1+\kappa-\tau^*)^{-\frac{\chi}{1-\chi}} \\
&\quad \times(-1+\alpha-(1-\tau^*)^2(-1+\chi)^2+(1-\tau^*)(-1+\chi)(-2+\alpha+\alpha\chi)) \\
&\quad +(-1+\alpha)\rho(1-\tau^*)\chi k(w^{\frac{1}{\chi}}(1-\tau^*)^{\frac{1}{(1-\chi)\chi}}(1+\kappa-\tau^*)^{-\frac{1}{1-\chi}}) \\
&\quad +(1-\tau^*)(-1+\alpha(3+\alpha(-3+\chi))-2(-1+\alpha)^2(1-\tau^*)(-1+\chi) \\
&\quad +(-1+\alpha)(1-\tau^*)^2(-1+\chi)^2)\chi k^{\alpha}(w^{\frac{1}{\chi}} \\
&\quad \times(1-\tau^*)^{\frac{1}{(1-\chi)\chi}}(1+\kappa-\tau^*)^{-\frac{1}{1-\chi}})))]/(1-\tau^*)\chi(\alpha-\tau^*-(1-\tau^*)\chi)^2.
\end{aligned}
\right.
$$
(38)

In (**??**), constants are respectively shown as follows: $k = k^* = (\frac{\alpha}{\rho})^{\frac{1}{1-\alpha}}$, $w = w^* = (1-\alpha)k^{\alpha}$, and $\kappa = \frac{\alpha}{1-\alpha}$, which refers to a ratio of capital in output to that of labor in output.

Based on the above relationships, we can examine what conditions cause (**??**) and (**??**). Since we can consider from (**??**) that both (**??**) and (**??**) are respectively a function of τ^*, we focus on how the shapes of the functions move according to changes in τ^*. In this context, τ^* is determined by the two intersections, where government spending equals the tax revenue. As a

result, two tax rates exist, one larger, another smaller, corresponding to the tax spending given. It is only with the two τ^*s determined in the above way that we consider (**??**) and (**??**). Naturally, in this situation, we assume that the tax spending given is changeable. With the above preparations, we can obtain expressions concerning $\theta_2 + \theta_3$ and $\theta_2\theta_3$, which are a function with τ^*, as follows:

$$
t(\tau^*) = \theta_2 + \theta_3 = \frac{1}{(\alpha + \tau^*(-1+\chi) - \chi)^2 \chi} \alpha \left(\frac{\alpha}{\rho}\right)^{\frac{1}{-1+\alpha}}
$$

$$
\left(-2(-1+\alpha)\left(\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha} (\tau^*)^3(-1+\chi)^3 -\right.
$$

$$
\left(\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha} (\tau^*)^2(-1+\chi)\left(2 + 2\alpha^2(-1+\chi) + \chi(-6+5\chi) + \alpha\left(2+\chi-4\chi^2\right)\right) +
$$

$$
\frac{-(-1+\alpha)\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \rho\chi + \left(\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha} \chi\left(-\alpha + \alpha(2+\alpha)\chi - (1+2\alpha)\chi^2 + \chi^3\right)}{-1+\chi} -
$$

$$
\tau^*\left((-1+\alpha)\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \rho\chi +\right.
$$

$$
\left.\left.\left(\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}\left(2\alpha - 2(1+\alpha)\chi + \left(3+\alpha-2\alpha^2\right)\chi^2 + 2(-1+\alpha)\chi^3\right)\right)\right),
$$

$$
\tag{39}
$$

and

$$d(\tau^*) = \frac{1}{(\alpha + \tau^*(-1+\chi) - \chi)^2(-1+\chi)^2}$$

$$\left((-1+\alpha)\left(\frac{\alpha}{\rho}\right)^{\frac{1}{-1+\alpha}}\rho\right.$$

$$\left(-\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\rho(1+\tau^*(-1+\chi)) + \alpha\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\rho(1+\tau^*(-1+\chi)) - \right.$$

$$\left(\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}(-1+\tau^*)^2(-1+\chi)^3(-\tau^* + (-1+\tau^*)\chi) + $$

$$(\alpha^2\left(\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}\left(-1+\tau^*(-1+\chi)\left(-1+\tau^*(-1+\chi)^2 + \chi - \chi^2\right)\right) + $$

$$\alpha\left(\left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}((\tau^*)^3(-1+\chi)^4 - (\tau^*)^2(-1+\chi)^3(1+2\chi) - (-2+\chi)(1+(-1+\chi)\chi) + $$

$$\tau^*\left(-3+\chi\left(7-5\chi+\chi^3\right)\right)\right)\right)\right).$$

(40)

At this stage, we need to examine how $t(\tau^*)$ and $d(\tau^*)$ behave. We obtain the following properties in relation to $t(\tau^*)$ and $d(\tau^*)$: $t(0) > 0$ and $t(1) = (2\rho\left(\alpha - 2\alpha\chi + \chi^2\right))/(1-\alpha)\chi < 0$; $d(0) < 0$, $d(1) = -\rho^2 < 0$ and $d(1) \lessgtr t(1)$.

In light of (??), it is obvious that $a_{21}(\tau^*)$, $a_{22}(\tau^*)$ and $a_{23}(\tau^*)$ have a common asymptote, $\tau^* = \frac{\alpha-\chi}{1-\chi}$. Therefore, at $\frac{\alpha-\chi}{1-\chi}$, $t(\tau^*)$ and $d(\tau^*)$ fall in discontinuity. In fact, they have the following properties: $\lim_{\tau^* \to \frac{\alpha-\chi}{1-\chi}-0} t(\tau^*) \to +\infty$ and $\lim_{\tau^* \to \frac{\alpha-\chi}{1-\chi}+0} t(\tau^*) \to -\infty$. On the other hand, $\lim_{\tau^* \to \frac{\alpha-\chi}{1-\chi}-0} d(\tau^*) \to -\infty$ and $\lim_{\tau^* \to \frac{\alpha-\chi}{1-\chi}+0} d(\tau^*) \to +\infty$.

Here, since we consider both $\theta_2 + \theta_3$ and $\theta_2\theta_3$ as a curve with an independent variable τ^*, we can easily speculate that the sign of $\theta_2 + \theta_3$ and $\theta_2\theta_3$ can change at this point where the asymptote holds. This implies a possibility that at this points in terms of τ^*, indeterminacy can begin to occur. At $\frac{\alpha-\chi}{1-\chi}$, our economic system begins to show indeterminacy if some conditions in relation to fundamental parameters such as α and χ are satisfied. Strictly speaking, ρ can also influence the sign. However, we omit this possibility, because we focus on labor supply, and ρ is not included in our labor supply function. In order to verify the above inference, let us visualize $\theta_2 + \theta_3$ and $\theta_2\theta_3$ in a figure. We obtain figure 2 as a phase digraph, where $\tau^{\star\star}$ is a tax

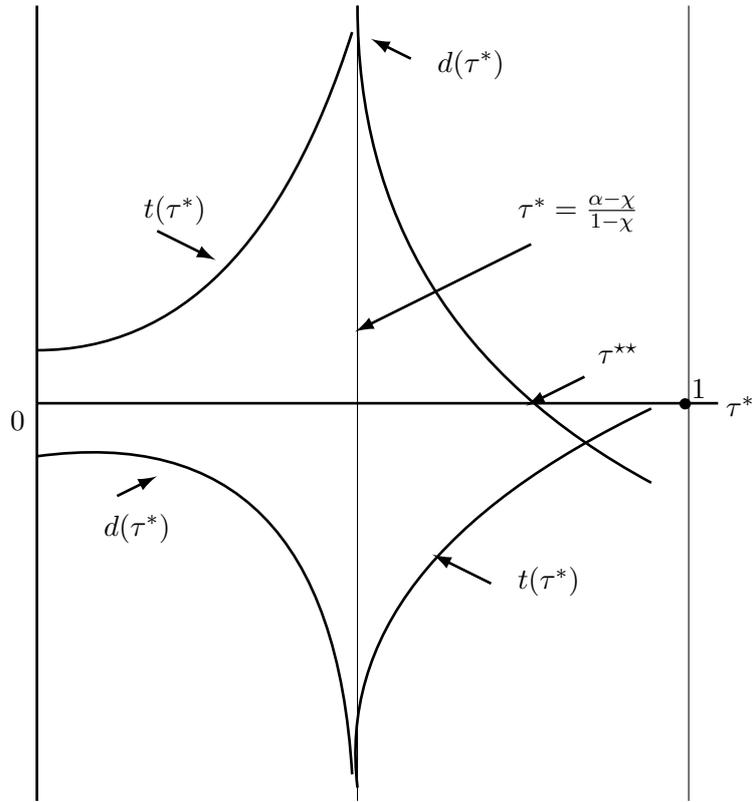rate corresponding to the maximum tax revenue.[14]

Figure 2: Given $\dfrac{\alpha - \chi}{1 - \chi} < \tau^* < \tau^{\star\star}$, indeterminacy paths exist; when $0 < \tau^* < \dfrac{\alpha - \chi}{1 - \chi}$, or $\tau^{\star\star} < \tau^* < 1$, a saddle path occurs.

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[14]The case of $d(1) < t(1)$ is described.

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We can see that at $\tau^* = \frac{\alpha-\chi}{1-\chi}$, the sign of $\theta_2 + \theta_3$ switches from positive to negative, and that of $\theta_2\theta_3$ switches from negative to positive, as τ^* moves from null to 1. And this shows that the movement pattern in our economy switches from a saddle path to indeterminacy. Of course, given $0 < \tau^* < \frac{\alpha-\chi}{1-\chi}$ or $\tau^* > \tau^{\star\star}$, saddle paths exist, while given $\frac{\alpha-\chi}{1-\chi} < \tau^* < \tau^{\star\star}$, indeterminacy exists.

At this stage, let us investigate indeterminacy from an economic viewpoint more intensively. Naturally, we should mainly focus on the labor supply. A representative household decides how much time it offers, anticipating future income tax rates, and the tax rates are finally decided in order to observe the balanced budget principle. First, suppose that the household anticipates a rise in the tax rates in the future. This anticipation causes a reduction in the labor supply in the future, which causes a reduction in the anticipated rate of return on capital, because of a rise in the ratio of capital to labor. And this reduction causes an increase in current consumption and a decrease in current labor supply. Second, the decrease in current labor supply leads to a decrease in the current tax base, namely labor income. This makes the government decide to impose higher tax rates in order to recover a balanced budget. In other words, the anticipation concerning a tax rise in the future can cause current high tax rates. Anticipation can be self-fulfilling. This is an interpretation in relation to indeterminacy.

However, this mechanism does not work unconditionally. First, this tax hike might not cause an increase in tax revenue. The tax revenue curve, which looks like a bell curve, implies that there is some range in which high tax rates cause a reduction in tax revenue. We know that this occurs given tax rate ranges over those corresponding to maximum revenue, that is, $\tau^{\star\star}$. Therefore, $\tau^{\star\star}$ is the largest tax rate involving indeterminacy. Moreover, strictly speaking, we can not say whether $\tau^{\star\star}$ is included in that range or not[15]. Second, for a self-fulfilling situation to occur, the current increase in tax rates corresponding to the expected increase in future taxes need not be large. Even in a range of comparatively low tax rates, this phenomenon is likely to occur[16]. This is why $\frac{\alpha-\chi}{1-\chi}$ is considered to be the minimum tax rate for indeterminacy.

In the next subsection, we explore what conditions determine $\frac{\alpha-\chi}{1-\chi} < \tau^{**}$.

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[15]Our model can not deal with this situation. This can applies to Schmitt-Grohé and Uribe (1997) as well.

[16]The same statement is made in Anagnostopoulous and Giannitsarou (2012).

7.1 An upper limitation

Based on the above analyses, it is obvious that for indeterminacy, $\frac{\alpha-\chi}{1-\chi}$ has to be less than $\tau^{\star\star}$. Therefore,

$$\frac{\chi(2-\alpha)-2+\sqrt{\alpha\left(\alpha\chi^2-4\chi+4\right)}}{2(1-\alpha)(\chi-1)} > \frac{\alpha-\chi}{1-\chi}$$

has to be satisfied. Calculating the above inequality, we obtain

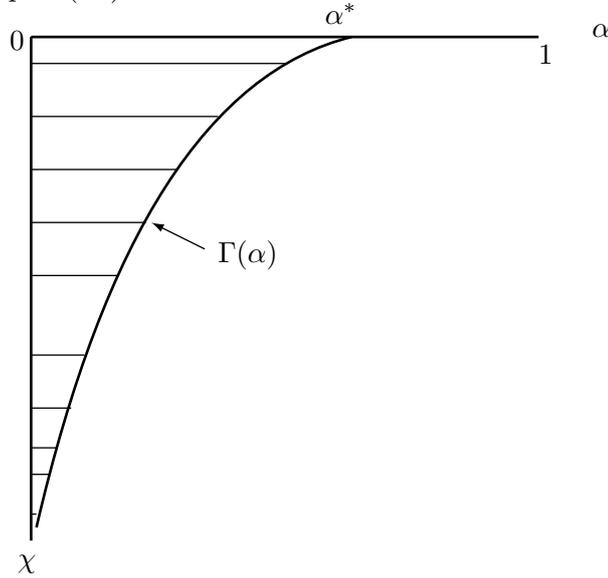$$0 > \chi > \frac{\alpha^3-\alpha^2+2\alpha-1}{\alpha^2} = \Gamma(\alpha). \tag{41}$$

Here, let us depict (**??**).

Figure 3: Combinations of α and χ for indeterminacy: when α and χ are located in the part with horizontal lines, indeterminacy occurs.

The above relationship shows that if a set { α, χ } satisfying (**??**) can exist, indeterminacy occurs. Additionally, we can consider that the right side in the above inequality is a function with an argument α, $\Gamma(\alpha)$. The function has properties such as an asymptote approaching negative infinity as α approaches null, increases monotonously in relation to α and is negative for $0 < \alpha < \alpha^*$ and positive for $\alpha^* < \alpha < 1$. In light of this diagram, we can consider the maximum value of α corresponding an arbitrary $\chi < 0$. We can verify that the maximum length between the two edges of τ^* is $1/(1-\chi)$,

which is when α is null. On the other hand, $\Gamma(\alpha)$ shows the length is null. So, once we choose one χ when α is null and horizontally increase α to $\Gamma(\alpha)$, the value of α at this intersection is the maximum value of α corresponding to the initial χ. In this way, we can quantitatively analyze the maximum values of α, if χ is given. We can say that the upper limitation with regard to α becomes small as χ declines.

In this situation, since χ is negative, α has an upper limitation, approximately $\alpha^* = 0.5698$. In short, this shows that in our economy with divisible labor, since we assume negative marginal disutility of labor, for indeterminacy to occur, the ratio of labor in output must be over 0.4302, and this value is independent of the values of other parameters.

In general, if an elasticity of labor supply with respect to wages, $-1/\chi$, is given, there is an upper limitation concerning α, which is shown by (??), in order for indeterminacy to occur. The expression *upper limitation concerning α* can be replaced by *concerning the ratio of capital in output to that of labor in output*. This explanation might be clearer than the one above from an economic viewpoint.

(??) can be transformed into the following

$$e = -\frac{1}{\chi} > \frac{-\alpha^2}{\alpha^3 - \alpha^2 + 2\alpha - 1} = \frac{\kappa^2(1 + \kappa)}{1 + \kappa - \kappa^3}. \tag{42}$$

In the above relationship, the function with an argument κ has properties such as an asymptote at $\kappa^* = 1.3247$, increasing monotonously in regard with κ, and being positive for $0 < \kappa < \kappa^*$, negative for $\kappa^* < \kappa < \infty$. Based on these properties, we can say that there is an upper limit concerning the ratio, κ. And the maximum ratio in indeterminacy is approximately 1.3247.

Hence we have a theorem:

Theorem 3 *When indeterminacy occurs, the ratio of capital in output to that of labor in output, κ, must be less than 1.3247 approximately.*

In general, the existence of indeterminacy depends on the value of α and χ. Let us take examples. If χ approaches minus infinity, both $(\alpha - \chi)/(1 - \chi)$ and τ^{**} respectively approach 1, which implies that there can be no indeterminacy, that is, all paths are saddles. This situation clearly occurs regardless of α. On the other hand, Figure 3 shows that when α is near 0.5698, χ can be null. This phenomenon means that both $(\alpha - \chi)/(1 - \chi)$ and τ^{**} are near 0.5698, which results in no indeterminacy. In the above situations, both $(\alpha - \chi)/(1 - \chi)$ and τ^{**} have the same values: 1 in the former case, and 0.5698 in the latter case.

For indeterminacy to occur with high frequency, obviously both α and χ need to be near null. In this situation, $(\alpha-\chi)/(1-\chi)$ becomes approximately null, while τ^{**} nears 1.

8   Conclusions

We have analyzed how our economy under a balanced budget rule moves.

In this paper, we postulated a utility function which features increasing marginal disutility with regard to labor supply, that is, a divisible one, while Schmitt-Grohé and Uribe, and their followers such as Anagnostopoulous and Giannitsarou, posit indivisible labor. Due to our postulation, we can specifically deduce a labor supply function, which can be affected by α and χ, and it becomes obvious that the shape of the tax revenue function is like a bell curve, with an independent argument regarding tax rates at steady states. With the help of global analysis, we demonstrate that this causes the existence of two steady states on the assumption of the balanced budget rule. Moreover, we demonstrate that in our economy, there is a maximum tax revenue. We have also demonstrated that at this maximum tax revenue, there is a double steady state. Thus, we conclude that with an evaluation of the linear approximation system in terms of a tax rate range, for a certain range of τ^*, our economy exhibits indeterminacy. In this context, the low edge is $(\alpha - \chi)/(1 - \chi)$, and the upper one is the crucial rate, which corresponds to the maximum tax revenue. Moreover, in order for this to hold, the ratio of capital in output to that of labor in output has to be less than 1.3247. On the other hand, at a higher tax rate than in this crucial range, and at a smaller tax rate range than $(\alpha - \chi)/(1 - \chi)$, our economy can converge to steady states along a saddle path.

These phenomena are caused by the assumption of the balanced budget principle. In other words, the existence of two steady states is caused by this principle. This is shown in our model by the fact that τ is not an independent variable, and that an eigenvalue from the characteristic polynomial $\phi(\theta)$ is null, which is deduced from Jacobian matrix A. Thus, τ plays a role as a kind of variable parameter. As a consequence, when G changes, τ^* can change, and the characteristic polynomial $\phi(\theta)$ can correspondingly shift. Then, since the eigenvalues can change, two movement patterns appear, namely a saddle path and indeterminate paths.

In this situation, the government can have no freedom with regard to tax rate policies. Regarding the saddle path, since the initial condition of a state variable $K(0)$ determines initial conditions in relation to adjoined variables,

there is no room for the government to intervene. But by arranging the scale of G, the government can avoid indeterminacy.

Our conclusion seems similar to that of Schmitt-Grohé and Uribe (1997) on appearance. But Schmitt-Grohé and Uribe (1997) deals with an economy with indivisible labor, which Hansen (1985) established. In such an economy, individual households simply choose to work or not. This assumption makes a representative household have a linear utility function with regard to labor. This leads to infinite elasticity of labor supply in relation to wages. Moreover, individual households face uncertainty concerning employment, and therefore the representative household aims to maximize its present value of expected utility. In addition, the expected utility concerning labor consists of two elements, which correspond to working or not. Obviously, the utility which occurs when the household does not work is null. In short, the expected utility in fact consists of one utility. This is why both systems resemble each other qualitatively. On the other hand, we deal with an economy without uncertainty concerning employment, which results in discussing numerical elasticity of labor supply in relation to wages. As a result, we were able to obtain Figure 3. According to Figure 3, it is obvious that there is no room for indeterminacy if χ approaches minus infinity, that is, the elasticity approaches null. Conversely, Schmitt-Grohé and Uribe (1997) only discusses an economy on the highest horizon line. In this sense, our perspective is wider and more realistic.

Appendix

Let us consider a function $\varphi(x)$, which was defined as

$$\varphi(x) = x[1 - \Omega(1-x)^{1-\chi}] + \frac{\alpha}{1-\alpha}.$$

First, we focus on the contents in the bracket above, here defined as $f(x) = 1 - \Omega(1-x)^{1-\chi}$. We can easily obtain the following information with regard to $f(x)$: $f'(x) = \Omega(1-\chi)(1-x)^{-\chi} > 0$, $f(0) = 1 - \Omega \gtrless 0$ and $f(1) = 1$.

Based on the above information, we can say that if $\Omega \leqq 1$, $f \geqq 0$, and that therefore, since $\varphi(x) > 0$, there is no steady state. So, we assume $\Omega > 1$

Differentiating $\varphi(x)$ with regard to x, we obtain the following:

$$\varphi'(x) = \Omega(1-x)^{-\chi}[(2-\chi)x - 1] + 1$$
$$= \Omega(1-x)^{-\chi}[(2-\chi)x - 1 + \frac{1}{\Omega}(1-x)^{\chi}].$$

Focusing on the contents in the second bracket in the above relationships, we investigate how $\varphi'(x)$ behaves. We define $\eta_1(x) = \left(\frac{1}{\Omega}\right)(1-x)^\chi$, which has properties such as $\eta_1'(x) = -\left(\frac{1}{\Omega}\right)\chi(1-x)^{\chi-1} > 0, \eta_1(0) = \frac{1}{\Omega} > 0$ and $\eta_1(1) = \infty$. On the other hand, we define $\eta_2(x) = 1 - (2-\chi)x$, which is monotonously decreasing with regard to x. Additionally, $\eta_2(0) = 1$ and $\eta_2(1) = \chi - 1 < 0$.

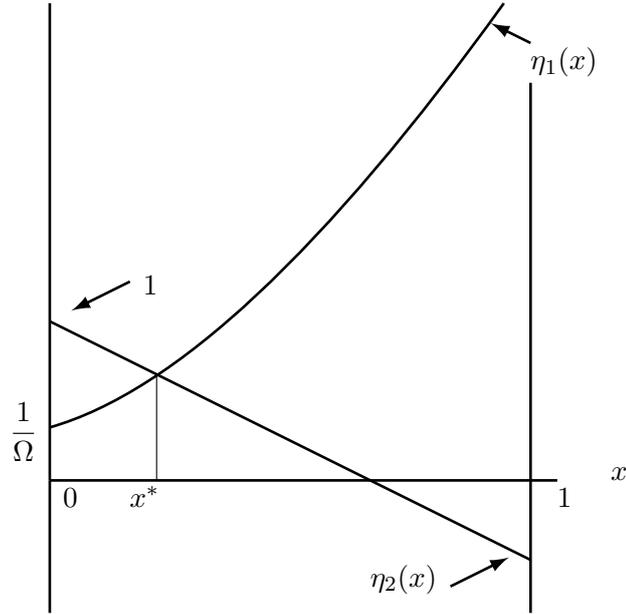Depicting the above relationships, we obtain the following:

Figure 4: $\mathbf{x}^*$ exists uniquely.

When G is given, x^* is naturally determined as satisfying $1 - (2 - \chi)x^* = \frac{1}{\Omega(1-x^*)^{-\chi}}$. From the above figure, it becomes obvious that $\varphi(x)$ monotonously decreases when $0 < x < x^*$, and increases when $x^* < x < 1$. Additionally, $\varphi(x)$ has properties such as $\varphi(0) = \alpha/(1-\alpha) > 0$ and $\varphi(1) = 1 + \alpha/(1-\alpha) > 0$.

Acknowledgements

Herrick. I note it with appreciation. Naturally, I accept full responsibility for any errors.

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