**Bertrand competition in vertically related markets**

**Tomomichi Mizuno**
**Kazuhiro Takauchi**

**April 2022**
**Discussion Paper No. 2208**

GRADUATE SCHOOL OF ECONOMICS

KOBE UNIVERSITY

ROKKO, KOBE, JAPAN

# Bertrand competition in vertically related markets

Tomomichi Mizuno*    Kazuhiro Takauchi†‡

April 12, 2022

**Abstract**

We build a successive Bertrand model with homogenous good. We show that increasing the production efficiency of upstream industry can reduce upstream firms' profits. We also show that increasing the production efficiency of downstream industry may reduce downstream firms' profits. Hence, an industrial policy that aims at improving production efficiency may be undesirable for firms.

**Key words:** Successive Bertrand; Production efficiency; Homogenous good

**JEL classification:** L13

*Graduate School of Economics, Kobe University, 2-1 Rokkodai-cho, Nada-ku, Kobe-City, Hyogo 657-8501, Japan. E-mail: mizuno@econ.kobe-u.ac.jp.

†Corresponding author: Kazuhiro Takauchi, Faculty of Business and Commerce, Kansai University, 3-3-35 Yamate-cho, Suita, Osaka 564-8680, Japan. E-mail: kazh.takauch@gmail.com; Tel.: +81-6-6368-1817.

‡Research Fellow, Graduate School of Economics, Kobe University.

# 1 Introduction

We consider Dastider (1995)-type homogenous-good Bertrand competition in both upstream and downstream markets. We show that when upstream and downstream production is efficient, increasing upstream and downstream production efficiency reduces upstream and downstream firms' profits, respectively. Because improving efficiency in a vertical structure does not necessarily increase firms' profits, our results imply that aiming to improve efficiency is a problematic issue in industrial policy.

According to an empirical study conducted by Flath (2012), among 70 Japanese industries, 35 engage in Bertrand competition, whereas five engage in Cournot competition. Although we assume homogenous-good Bertrand competition in both upstream and downstream markets, empirical evidence suggests that price competition is more common than quantity competition; hence, our model is partially consistent with the results of the empirical study.

We incorporate Dastider (1995)-type Bertrand competition into a Salinger (1988)-type vertical structure model. Salinger (1988) assumed Cournot competition in both upstream and downstream markets.[1] Dastider (1995) examined homogenous Bertrand competition with a convex cost of firms, and demonstrated that, in homogenous Bertrand competition, equilibrium has a range if oligopolists have a convex cost.[2] To the best of our knowledge, price competition for a homogeneous product in both upstream and downstream markets has not been considered in a study.

# 2 Model

We consider a market with $m$ upstream and $n$ downstream firms. Each upstream and downstream firm produces homogenous input and a final product, respectively. Upstream firms

[1] Salinger's (1988) model is frequently used in industrial organization literature, for example, see Matsushima (2006) and Mukherjee (2019).

[2] Dastider's model (1995) is applied in various scenarios; see, for example, Cabon-Dhersin and Drouhin (2014), Mizuno and Takauchi (2020), and Takauchi and Mizuno (2021).

have a convex cost function; we assume that the cost is $\gamma x_k^2$ if the output of upstream firm $k$ $(= 1, \ldots, m)$ is $x_k$. Upstream firms compete on price, and upstream firm $k$'s input price is denoted by $w_k$. Then, upstream firm $k$'s profit is $\pi_k^u \equiv w_k x_k - \gamma x_k^2$.

Each downstream firm purchases input from upstream firms, choosing the lowest input price $w \equiv \min\{w_1, \ldots, w_m\}$. If multiple upstream firms choose $w$, then downstream firms purchase an equal quantity of inputs from upstream firms. We assume that to produce one unit of output, each downstream firm uses one unit of input. Additionally, downstream firms incur production costs. When downstream firm $i$ $(= 1, \ldots, n)$ produces $q_i$ output, its production cost is $\lambda q_i^2$. Downstream firms compete on price and the price of downstream firm $i$ is denoted by $p_i$. The demand function in the downstream market is $Q \equiv (a - p)/b$, where $p \equiv \min\{p_1, \ldots, p_n\}$. Each downstream firm's demand is determined by $q_i = Q$ if for any $j$ $(= 1, \ldots, n)$ and $i \neq j$, $p_i < p_j$; $q_i = Q/s$ if $p_i$ is the lowest price and $s$ downstream firms choose it; and $q_i = 0$ if $p_i$ is not the lowest price. Downstream firm $i$'s profit is $\Pi_i^d \equiv (p_i - w)q_i - \lambda q_i^2$. The consumer surplus is $CS = bQ^2/2$ and the total surplus is $SW = CS + \sum_{k=1}^{m} \pi_k^u + \sum_{j=1}^{n} \Pi_j^d$.

The timing of the game is as follows: In the first stage, each upstream firm chooses its input price $w_k$. In the second stage, each downstream firm decides where to purchase its input and chooses its final-good price $p_i$.

Because we assume homogenous price competition with a convex cost, the equilibria of each stage game are given by a certain closed interval (Dastider, 1995). To choose a unique equilibrium, we use the payoff-dominance criterion as equilibrium refinement. This criterion was often used in previous studies (e.g., Cabon-Dhersin and Drouhin, 2014; Mizuno and Takauchi, 2020). Hence, the equilibrium concept is a subgame-perfect equilibrium with a payoff-dominance criterion. We further assume that $\lambda < bn/(n-1) \equiv \hat{\lambda}$. Under this assumption, the downstream price chosen through equilibrium refinement with the payoff-dominance criterion does not coincide with the price that maximizes the joint profits of downstream firms.

# 3 Equilibrium calculation

Because downstream firms are symmetric, they choose symmetric price $p(w)$ in equilibrium. We define the aggregate and each downstream firm's outputs at $p(w)$ as $Q(w) = [a - p(w)]/b$ and $q(w) = Q(w)/n$, respectively. In equilibrium, downstream firms never deviate from $p(w)$. Hence, the following inequalities must be satisfied:

$$[p(w) - w]q(w) - \lambda q(w)^2 \geq [p(w) - w]Q(w) - \lambda Q(w)^2, \quad [p(w) - w]q(w) - \lambda q(w)^2 \geq 0.$$

The first inequality represents the condition under which each downstream firm does not undercut the price and the second inequality indicates that each downstream firm has no incentive to increase its price.

Solving the above inequalities for $p(w)$, we obtain Nash equilibrium prices as the interval: $p(w) \in [\underline{p}(w), \bar{p}(w)]$, where

$$\underline{p}(w) = \frac{bnw + a\lambda}{bn + \lambda}, \quad \bar{p}(w) = \frac{bnw + a(1+n)\lambda}{bn + (1+n)\lambda}. \tag{1}$$

Because we use the payoff-dominance criterion as equilibrium refinement, we derive symmetric collusive price $p_{col}(w)$ that maximizes the joint profit of downstream firms $\sum_{j=1}^{n} \Pi_j^d$:

$$p_{col}(w) = \frac{a(bn + 2\lambda) + bnw}{2(bn + \lambda)}. \tag{2}$$

Under the assumption $\lambda < \hat{\lambda}$, we obtain $\bar{p}(w) < p_{col}(w)$. Hence, the subgame outcome is $\bar{p}(w)$; that is, downstream firms choose the highest price in Nash equilibria.

Next, we consider the first stage. Because upstream firms are symmetric, they choose symmetric input price $w^*$ in equilibrium, which leads to symmetric demand $Q(w^*)/m$. As in the second stage, the conditions under which upstream firms do not undercut or raise prices are expressed by the following inequalities:

$$w^* \frac{Q(w^*)}{m} - \gamma \left[\frac{Q(w^*)}{m}\right]^2 \geq w^* Q(w^*) - \gamma Q(w^*)^2, \quad w^* \frac{Q(w^*)}{m} - \gamma \left[\frac{Q(w^*)}{m}\right]^2 \geq 0.$$

Then, input prices in Nash equilibria are given by the interval $w^* \in [\underline{w}, \bar{w}]$, where

$$\underline{w} = \frac{a\gamma n}{bmn + \lambda m + \lambda mn + \gamma n}, \quad \bar{w} = \frac{a\gamma(m+1)n}{bmn + \lambda m + n(\gamma + \gamma m + \lambda m)}.$$

Similarly, the price that maximizes the joint profit of upstream firms is

$$w_{col} = \frac{a(bmn + \lambda m + \lambda mn + 2\gamma n)}{2[bmn + \lambda m + n(\gamma + \lambda m)]}.$$

Comparing the above input prices, we obtain the following lemma.

**Lemma 1.** *(i)* $\bar{w} > \underline{w}$*. (ii)* $w_{col} > \bar{w}$ *iff* $\gamma < \frac{m[bn+(1+n)\lambda]}{n(m-1)} \equiv \hat{\gamma}$.

*Proof.* (i) $\bar{w} - \underline{w} = \frac{a\gamma m^2 n(bn+\lambda+\lambda n)}{(bmn+\lambda m+\lambda mn+\gamma n)(bmn+\lambda m+\gamma mn+\lambda mn+\gamma n)} > 0$.

(ii) $w_{col} - \bar{w} = \frac{am(bn+\lambda+\lambda n)(bmn+\lambda m-\gamma mn+\lambda mn+\gamma n)}{2(bmn+\lambda m+\lambda mn+\gamma n)(bmn+\lambda m+\gamma mn+\lambda mn+\gamma n)}$. Hence, Lemma 1 holds. □

From this lemma, the equilibrium input price is $\bar{w}$ if $\gamma < \hat{\gamma}$; that is, $w_{col}$ if $\gamma \geq \hat{\gamma}$.

# 4 Comparative statics

## 4.1 Case with upper bound pricing

First, we consider the case with $\gamma < \hat{\gamma}$; that is, equilibrium upstream and downstream prices are $w = \bar{w}$ and $p = \bar{p}(\bar{w})$. Substituting these prices into the upstream and downstream profits yields

$$\bar{\pi}^u = \frac{a^2\gamma mn^2}{[bmn + \lambda m + n(\gamma + \gamma m + \lambda m)]^2}, \tag{3}$$

$$\bar{\Pi}^d = \frac{a^2\lambda m^2 n}{[bmn + \lambda m + n(\gamma + \gamma m + \lambda m)]^2}. \tag{4}$$

From (3) and (4), we establish Proposition 1.[3]

**Proposition 1.** *Suppose* $\gamma < \hat{\gamma}$*. (i) The profit of each upstream firm is a single-peaked function of* $\gamma$ *and takes its maximum value at* $\gamma = m[bn + (1+n)\lambda]/[n(m+1)] \equiv \bar{\gamma}$*; the profit of*

[3]For welfare analysis, we only have well-known results. Hence, these results are reported in the Online Appendix.

*each upstream firm decreases with $\lambda$. (ii) The profit of each downstream firm is a single-peaked function of $\lambda$ and takes its maximum value at $\lambda = n[bm + (1+m)\gamma]/[m(n+1)] \equiv \bar{\lambda}$; the profit of each downstream firm decreases with $\gamma$.*

*Proof.* See the Appendix.

From this proposition, we confirm that it is possible that efficient technology reduces each firm's profit. The intuition behind this result is as follows.

First, we consider the effects of production efficiency in upstream and downstream markets on upstream firms' profits. The inefficiency of upstream technology, that is, an increase in $\gamma$, has two effects: production inefficiency effect and competition mitigation effect. Because the production cost of upstream firms is $\gamma x_k^2$, large $\gamma$ leads to inefficient production. Additionally, the equilibrium input price is determined by the condition that each upstream firm does not undercut the equilibrium input price. For large $\gamma$, if an upstream firm gains whole-market demand by undercutting the input price, its production cost also becomes large. Therefore, when $\gamma$ is large, the equilibrium input price increases because upstream firms are less likely to undercut the input price. Thus, an increase in $\gamma$ has the effect of relaxing competition.

Next, consider the scenario in which each of the effects dominates. As $\gamma$ converges to zero, upstream competition approaches Bertrand competition with a constant marginal cost. Then, upstream firms' profits converge to zero. Additionally, when $\gamma$ diverges infinitely, the profits of upstream firms converge to zero. Thus, when $\gamma$ is small, the competition mitigation effect dominates, and when $\gamma$ is large, the production inefficiency effect dominates. Therefore, the profit of each upstream firm is an inverted-U shape for $\gamma$.

When downstream production becomes less efficient, that is, an increase in $\lambda$, upstream firms face less efficient trading partners. Hence, an increase in $\lambda$ always decreases upstream firms' profits.

Second, we consider the effects of production efficiency in upstream and downstream markets on downstream firms' profits. These effects are similar to those on upstream firms' profits. An increase in $\lambda$ has production inefficiency and competition mitigation effects. Hence, the profit of each downstream firm is an inverted-U shape for $\lambda$. Additionally, for large $\gamma$, downstream firms face inefficient input suppliers. Hence, downstream firms' profits decrease with $\gamma$.

Proposition 1 directly leads to the following.

**Corollary 1.** *Upstream and downstream efficiency levels that maximize upstream and downstream profits increase with $\lambda$ and $\gamma$, respectively; that is, $\partial\bar{\gamma}/\partial\lambda > 0$ and $\partial\bar{\lambda}/\partial\gamma > 0$.*

As explained in the intuition behind Proposition 1, an increase in $\gamma$ has a production inefficiency effect and competition mitigation effect in the upstream market. These effects are balanced at $\gamma = \bar{\gamma}$, where upstream firms' profits are maximized. When downstream market inefficiency increases, that is, $\lambda$ increases, upstream firms can increase their profits by choosing a higher price that achieves less output; that is, as downstream market inefficiency increases, the competition mitigation effect is strengthened. Thus, at larger $\gamma$, upstream firms' profits are maximized, which means that $\bar{\gamma}$ increases with $\lambda$.

The reason that $\bar{\lambda}$ increases with $\gamma$ can be explained in the same manner. Consider the case in which $\lambda$ increases at $\gamma = \bar{\gamma}$, where downstream firms' profits are maximized. Then, $\gamma$ that maximizes downstream firms' profits increases because the effect of increasing $\gamma$ on mitigating competition is strengthened.

## 4.2 Case with upstream collusive pricing

Next, we examine the case with $\gamma \geq \hat{\gamma}$, which yields collusive pricing in the upstream market.[4] Then, the equilibrium prices are $w = w_{col}$ and $p = \bar{p}(w_{col})$, and equilibrium profits are

$$\pi^u_{col} = \frac{a^2 n}{4[bmn + \lambda m + n(\gamma + \lambda m)]}, \tag{5}$$

$$\Pi^d_{col} = \frac{a^2 \lambda m^2 n}{4[bmn + \lambda m + n(\gamma + \lambda m)]^2}. \tag{6}$$

(5) and (6) yield Proposition 2.

**Proposition 2.** *Suppose $\gamma \geq \hat{\gamma}$. (i) The profit of each upstream firm decreases with $\gamma$ and $\lambda$. (ii) The profit of each downstream firm is a single-peaked function of $\lambda$ and takes its maximum value at $\lambda = n(bm + \gamma)/[m(n+1)] \equiv \lambda_{col}$; the profit of each downstream firm decreases with $\gamma$.*

*Proof.* See the Appendix.

In this case, only the conditions that determine the equilibrium input price differ from the case with upper bound pricing. When the equilibrium input price is collusive, the competition mitigation effect of increased production inefficiency in the upstream market disappears. Hence, upstream firms' profits decrease with $\gamma$. The other results are similar to Proposition 1, and so are their intuitions.


## Acknowledgements

This study was supported by JSPS KAKENHI (20K01678, 19H0148, 20K01646). All errors are our own.


[4] For a more detailed argument on collusive pricing, for example, see Dastider (2001) and Takauchi and Mizuno (2021).

## Appendix A. Proofs

**Proof of Proposition 1**. Differentiating the profit of an upstream firm with respect to $\gamma$ or $\lambda$ yields

$$\frac{\partial\bar{\pi}^u}{\partial\gamma} = \frac{a^2mn^2[m(bn+\lambda+n\lambda)-n(1+m)\gamma]}{[bmn+\lambda m+n(\gamma+\gamma m+\lambda m)]^3}, \quad \frac{\partial\bar{\pi}^u}{\partial\lambda} = -\frac{2a^2\gamma m^2n^2(n+1)}{[bmn+\lambda m+n(\gamma+\gamma m+\lambda m)]^3} < 0.$$

Hence, $\partial\bar{\pi}^u/\partial\gamma > 0$ if $\gamma < \frac{m[bn+(1+n)\lambda]}{n(m+1)} \equiv \bar{\gamma}$. This threshold is undoubtedly smaller than $\hat{\gamma}$: $\bar{\gamma} < \hat{\gamma}$. Additionally, $\partial\bar{\pi}^u/\partial\gamma \leq 0$ if $\bar{\gamma} \leq \gamma < \hat{\gamma}$. Thus, $\bar{\pi}^u$ is a single-peaked function of $\gamma$.

Differentiating the profit of the downstream firm with respect to $\gamma$ or $\lambda$ yields

$$\frac{\partial\bar{\Pi}^d}{\partial\gamma} = -\frac{2a^2\lambda m^2(m+1)n^2}{[bmn+\lambda m+n(\gamma+\gamma m+\lambda m)]^3} < 0, \quad \frac{\partial\bar{\Pi}^d}{\partial\lambda} = \frac{a^2m^2n[n(bm+\gamma+m\gamma)-m(1+n)\lambda]}{[bmn+\lambda m+n(\gamma+\gamma m+\lambda m)]^3}.$$

Hence, $\partial\bar{\Pi}^d/\partial\lambda > 0$ if $\lambda < n[bm+(1+m)\gamma]/[m(n+1)] \equiv \bar{\lambda}$. Hence, $\bar{\lambda} < \hat{\lambda}$. Additionally, $\partial\bar{\Pi}^d/\partial\lambda \leq 0$ if $\lambda \geq \bar{\lambda}$. Therefore, $\bar{\Pi}^d$ is a single-peaked function of $\lambda$. □

**Proof of Proposition 2**. Differentiating profits yields

$$\frac{\partial\pi^u_{col}}{\partial\gamma} = -\frac{a^2n^2}{4[bmn+\lambda m+n(\gamma+\lambda m)]^2} < 0, \quad \frac{\partial\pi^u_{col}}{\partial\lambda} = -\frac{a^2mn(n+1)}{4[bmn+\lambda m+n(\gamma+\lambda m)]^2} < 0,$$
$$\frac{\partial\Pi^d_{col}}{\partial\gamma} = -\frac{a^2\lambda m^2n^2}{2[bmn+\lambda m+n(\gamma+\lambda m)]^3} < 0, \quad \frac{\partial\Pi^d_{col}}{\partial\lambda} = \frac{a^2m^2n[n(bm+\gamma)-m(1+n)\lambda]}{4[bmn+\lambda m+n(\gamma+\lambda m)]^3}.$$

Hence, $\partial\Pi^d_{col}/\partial\lambda > 0$ if $\lambda < n(bm+\gamma)/[m(1+n)] \equiv \lambda_{col}$. Additionally, $\partial\Pi^d_{col}/\partial\lambda \leq 0$ if $\lambda \geq \lambda_{col}$. Therefore, $\Pi^d_{col}$ is a single-peaked function of $\lambda$ and it takes its maximum value at $\lambda = \lambda_{col}$. □

## References


[1] Cabon-Dhersin, M. L., Drouhin, N., 2014. Tacit collusion in a one-shot game of price competition with soft capacity constraints. J. Econ. Manage. Strat. 23, 427-442.

[2] Dastider, K. G., 1995. On the existence of pure strategy Bertrand equilibrium. Econ. Theor. 5, 19-32.

[3] Dastider, K. G., 2001. Collusive outcomes in price competition. J. Econ. 73, 81-93.

[4] Flath, D., 2012. Are there any Cournot industries in Japan? Jpn. Econ. 35, 3-36.

[5] Matsushima, N., 2006. Industry profits and free entry in input markets. Econom. Lett. 93, 329-336.

[6] Mizuno, T., Takauchi, K., 2020. Optimal export policy with upstream price competition. Manch. Sch. 88, 324-348.

[7] Mukherjee, A., 2019. Profit raising entry in a vertical structure. Econom. Lett. 183, 108543.

[8] Salinger, M. A., 1988. Vertical mergers and market foreclosure. Q. J. Econ. 103, 345-356.

[9] Takauchi, K., Mizuno, T., 2021. Endogenous transport price, R&D spillovers, and trade. World Econ. (forthcoming)

# Online Appendix of "Bertrand competition in vertically related markets".

**by Tomomichi Mizuno and Kazuhiro Takauchi**

This appendix presents results of comparative statics of our model. For example, a change in firm numbers, $m$ and $n$, and production efficiencies, $\gamma$ and $\lambda$, brings standard and well-known welfare results.

**Input and final-good prices:** The differentiation with respect to firm numbers is:

$$\frac{\partial \bar{w}}{\partial m} = -\frac{a\gamma n(bn+\lambda+\lambda n)}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^2} < 0, \quad \frac{\partial \bar{p}}{\partial m} = -\frac{ab\gamma n^2}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^2} < 0,$$
$$\frac{\partial \bar{w}}{\partial n} = \frac{a\gamma\lambda m(m+1)}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^2} > 0, \quad \frac{\partial \bar{p}}{\partial n} = -\frac{ab\lambda m^2}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^2} < 0.$$

The differentiation with respect to $\gamma$ and $\lambda$ is:

$$\frac{\partial \bar{w}}{\partial \gamma} = \frac{am(m+1)n(bn+\lambda+\lambda n)}{[bmn+\lambda m+n(\gamma+\gamma m+\lambda m)]^2} > 0, \quad \frac{\partial \bar{p}}{\partial \gamma} = \frac{abm(m+1)n^2}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^2} > 0,$$
$$\frac{\partial \bar{w}}{\partial \lambda} = -\frac{a\gamma m(m+1)n(n+1)}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^2} < 0, \quad \frac{\partial \bar{p}}{\partial \lambda} = \frac{abm^2n(n+1)}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^2} > 0.$$

**Profits:** The differentiation with respect to firm numbers is:

$$\frac{\partial \bar{\pi}_k^u}{\partial m} = -\frac{a^2\gamma n^2(bmn+\lambda m+n(-\gamma+\gamma m+\lambda m))}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} < 0, \quad \frac{\partial \bar{\Pi}_i^d}{\partial m} = \frac{2a^2\gamma\lambda mn^2}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} > 0,$$
$$\frac{\partial \bar{\pi}_k^u}{\partial n} = \frac{2a^2\gamma\lambda m^2 n}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} > 0, \quad \frac{\partial \bar{\Pi}_i^d}{\partial n} = -\frac{a^2\lambda m^2(bmn-\lambda m+n(\gamma+\gamma m+\lambda m))}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} < 0.$$

**Welfare:** Consumer surplus, $\bar{CS}$, and total surplus, $\bar{SW}$, become:

$$\bar{CS} = \frac{a^2bm^2n^2}{2[bmn+\lambda m+n(\gamma+\gamma m+\lambda m)]^2}, \quad \bar{SW} = \frac{a^2m^2n^2(b+2(\gamma+\lambda))}{2[bmn+\lambda m+n(\gamma+\gamma m+\lambda m)]^2}.$$

The differentiation with respect to firm numbers is:

$$\frac{\partial \bar{CS}}{\partial m} = \frac{a^2b\gamma mn^3}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} > 0, \quad \frac{\partial \bar{SW}}{\partial m} = \frac{a^2\gamma mn^3(b+2(\gamma+\lambda))}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} > 0,$$
$$\frac{\partial \bar{CS}}{\partial n} = \frac{a^2b\lambda m^3 n}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} > 0, \quad \frac{\partial \bar{SW}}{\partial n} = \frac{a^2\lambda m^3 n(b+2(\gamma+\lambda))}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} > 0.$$

The differentiation with respect to $\gamma$ and $\lambda$ is:

$$\frac{\partial \bar{CS}}{\partial \gamma} = -\frac{a^2 b m^2 (m+1) n^3}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} < 0, \quad \frac{\partial \bar{SW}}{\partial \gamma} = -\frac{a^2 m^2 n^2 (bn-\lambda m+n(\gamma+2\lambda+\gamma m+\lambda m))}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} < 0,$$
$$\frac{\partial \bar{CS}}{\partial \lambda} = -\frac{a^2 b m^3 n^2 (n+1)}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} < 0, \quad \frac{\partial \bar{SW}}{\partial \lambda} = -\frac{a^2 m^2 n^2 (bm+m(2\gamma+\lambda+\gamma n+\lambda n)-\gamma n)}{(bmn+\lambda m+n(\gamma+\gamma m+\lambda m))^3} < 0.$$

## Upstream collusive pricing

**Profits:** The differentiation with respect to $m$ and $n$ is:

$$\frac{\partial \pi^u_{col}}{\partial m} = -\frac{a^2 n(bn+\lambda)}{2(2bmn+2\lambda m+\gamma n)^2} < 0, \quad \frac{\partial \Pi^d_{col}}{\partial m} = \frac{a^2 \gamma mn(bn+\lambda)}{2(2bmn+2\lambda m+\gamma n)^3} > 0,$$
$$\frac{\partial \pi^u_{col}}{\partial n} = \frac{a^2 \lambda m}{2(2bmn+2\lambda m+\gamma n)^2} > 0, \quad \frac{\partial \Pi^d_{col}}{\partial n} = -\frac{a^2 m^2 \left(2b^2 mn+2b\lambda m+b\gamma n+2\gamma\lambda\right)}{4(2bmn+2\lambda m+\gamma n)^3} < 0.$$

**Welfare:** Consumer surplus, $CS_{col}$, and total surplus, $SW_{col}$, become:

$$CS_{col} = \frac{a^2 b m^2 n^2}{8(2bmn+2\lambda m+\gamma n)^2}, \quad SW_{col} = \frac{a^2 mn(7bmn+6\lambda m+2\gamma n)}{8(2bmn+2\lambda m+\gamma n)^2}.$$

The differentiation with respect to firm numbers is:

$$\frac{\partial CS_{col}}{\partial m} = \frac{a^2 b\gamma mn^3}{4(2bmn+2\lambda m+\gamma n)^3} > 0, \quad \frac{\partial SW_{col}}{\partial m} = \frac{a^2 \gamma n^2(5bmn+4\lambda m+\gamma n)}{4(2bmn+2\lambda m+\gamma n)^3} > 0,$$
$$\frac{\partial CS_{col}}{\partial n} = \frac{a^2 b\lambda m^3 n}{2(2bmn+2\lambda m+\gamma n)^3} > 0, \quad \frac{\partial SW_{col}}{\partial n} = \frac{a^2 \lambda m^2(8bmn+6\lambda m+\gamma n)}{4(2bmn+2\lambda m+\gamma n)^3} > 0.$$

The differentiation with respect to $\gamma$ and $\lambda$ is:

$$\frac{\partial CS_{col}}{\partial \gamma} = -\frac{a^2 b m^2 n^3}{4(2bmn+2\lambda m+\gamma n)^3} < 0, \quad \frac{\partial SW_{col}}{\partial \gamma} = -\frac{a^2 mn^2(5bmn+4\lambda m+\gamma n)}{4(2bmn+2\lambda m+\gamma n)^3} < 0,$$
$$\frac{\partial CS_{col}}{\partial \lambda} = -\frac{a^2 b m^3 n^2}{2(2bmn+2\lambda m+\gamma n)^3} < 0, \quad \frac{\partial SW_{col}}{\partial \lambda} = -\frac{a^2 m^2 n(8bmn+6\lambda m+\gamma n)}{4(2bmn+2\lambda m+\gamma n)^3} < 0.$$