Intertemporal elasticity of substitution and the transitional dynamics and steady state of wealth distribution

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by

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Abstract

Although the steady state equilibrium is represented by a single point in the capital-consumption plane in the standard Ramsey model, it is by a straight line in a Ramsey model with heterogeneous individuals. Taking advantage of this fact, this paper applies the backward induction method to analyze the transitional dynamics of the Ramsey model with heterogeneous individuals, and examines the role of heterogeneity in intertemporal elasticity of substitution (IES). When no heterogeneity exists in IES across individuals, then the wealth Gini declines as capital accumulates, while the wealth gap expands. In contrast, with heterogeneity, various dynamics of wealth distribution can emerge, including a U-shaped relationship between income and inequality. It is also shown that an inverted U-shaped relationship, i.e., the Kuznets curve can be explained by Stone-Geary preferences, which allow IES to change with wealth.

JEL classification: C63; D31; D90; E21; O41

Keywords: Ramsey model; intertemporal elasticity of substitution; wealth distribution; transitional dynamics; Stone-Geary preference

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1. Introduction

Individual preferences, which are in essence characterized by time preference and intertemporal substitution, play a crucial role in determining the dynamics of wealth distribution and the distribution in the steady state. The most well-known result about the wealth distribution in modern growth theory is probably the Ramsey conjecture, which is first proposed by Ramsey (1928), and then confirmed by Becker (1980), and Mitra and Sorger (2013), among others, that the most patient individual, i.e., the individual whose time-preference rate is the lowest, will eventually have all the wealth in the economy. The conjecture of course refers not to the dynamics of wealth distribution but to the long-run phenomena.[1] To analyze the dynamics precisely, we cannot avoid the aggregation issue.[2]

If the utility function is homogeneous with respect to the relevant arguments, then the aggregate behavior can be considered as generated by a single representative agent.[3] Time-separable utility functions with a constant intertemporal elasticity of substitution (IES) that are most widely used in contemporary growth theory have this homogeneity property. Since the aggregate behavior of the economy is independent of the individuals' behavior, the analysis becomes tractable. Taking advantage of this fact, Caselli and Ventura (2000) and Garcı´a-Peñalosa and Turnovsky (2010), among others, analyze distributional issues to obtain the insightful results.

To apply the above strategy, IES must be the same across individuals. However, a large empirical literature, which includes Attanasio et al. (2002), Blundell et al. (2014) and Browning et al. (1995), points out heterogeneity in IES. Nevertheless, probably due to the analytical difficulty, enough attention may not have been paid to heterogeneity in IES in the determination of wealth distribution. Using a simple Ramsey model, this paper examines the role of heterogeneity on the wealth distribution not only in the steady state but also in the transitional dynamics.

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[1] Heterogeneity in the time preference across individuals is important also in determining transitional dynamics of wealth. See, for example, Fisher (2017).

[2] Of course, the aggregation is crucial not only for the dynamics but also for the static analysis when we build a macroeconomic model based on rigorous microfoundations.

[3] This important finding dates back to the seminal work by Gorman (1953).

The long-run steady state equilibrium is shown by a single point in the (k,c) plane in the standard Ramsey model with homogeneous individuals, where k stands for per-capita capital, while c for per-capita consumption. In contrast, as Nakamura (2020) shows, it is represented by a straight line in the (k,c) plane in the Ramsey model where individuals have a common rate of time preference but different IES. Although the location and slope of the line do not depend on the initial condition and heterogeneity in IES, the length depends crucially on them. Putting it differently, although the initial condition and individuals' IES are required to determine the line precisely, the basic nature of the long-run steady state equilibrium can be characterized independently of them. By utilizing that this fact allows us to apply the backward induction method to analyze the transitional dynamics of the Ramsey model with heterogeneous individuals, this paper examines the role of heterogeneity in IES on the dynamics of wealth distribution.

The empirical literature not only points out heterogeneity in IES but also finds that it increases with wealth, i.e., the rich have high IES, while the poor have low IES. As Guvenen (2006) clearly explains, this finding is very important to reconcile conflicting evidence on IES in business cycles. It is also crucial to understand the dynamics of wealth distribution and the distribution in the long-run steady state. Here it should be noted that wealth is accumulated net savings. The empirical finding that IES increases with wealth therefore implies that individuals with high IES accumulate wealth more rapidly than individuals with low IES. As a result, those with high IES have larger wealth than the others. This paper shows that this is true in the dynamic general equilibrium. Moreover, the possibility is shown that the relatively poor individuals in the initial period becomes relatively rich in the long-run steady state if their IES are high enough. Hence, we have an inverted Kuznets curve, that is, a U-shaped relationship between income and inequality.

To clarify the role of heterogeneity in IES, we treat it as exogenous variable in the most part of this paper. However, the individual IES may depend on wealth and/or consumption, i.e., may change along the transition. For example, the introduction of minimum consumption into individual preference allows IES to changes over the course of economic growth. Alvarez-Pelaez and Diaz (2005) and Obiols-Homs and Urrutia (2005) analyze this situation to obtain interesting results about transitional dynamics in wealth distribution. As an extension, we also analyze the Ramsey model with a Stone-Geary utility function to show that an inverted U-shaped relationship between income and inequality, i.e., the Kuznets curve can emerge. The results obtained from the model with heterogeneity help intuitively understand the mechanism behind the inverted U-shaped relationship.

The rest of the paper is organized as follows. Section 2 sets up the model and derives a couple of important theoretical results that allow us to conduct the simulation analysis. Section 3 deals with numerical simulations on the dynamics of wealth distribution to clarify the role of heterogeneity in IES. It also simulates the model with a Stone-Geary utility function to show that the Kuznets curve can emerge. Section 4 concludes the paper.

2. The model

Consider an infinite horizon economy that consists of many infinitely-lived individuals and identical firms. The individuals are divided into two types, P and R; all individuals of each type are identical in terms of both wealth and preference, while both wealth and preference can differ across the types.[4] The population of individuals is constant over time and normalized to unity. Denoting the population of type i individuals as λ_i ($0<\lambda_i<1$),

$$\lambda_P + \lambda_R = 1. \tag{1}$$

2-1. Technology and factor prices

Firms produce a homogeneous good according to the following standard constant-returns-to-scale production function $F(\cdot)$:

$$Y(t) = F(K(t), L(t)) \tag{2}$$

where $Y(t)$ is output, $K(t)$ is capital, and $L(t)$ is labor. The above can be rewritten as the following per-capita production function:

$$y(t) = f(k(t)) \quad \text{with} \quad f'(k(t)) > 0 \quad \text{and} \quad f''(k(t)) < 0, \tag{3}$$

where $y(t) = Y(t)/L(t)$ and $k(t) = K(t)/L(t)$. In equilibrium, the net rate of return on capital

[4] The main results remain unchanged if we assume more than two types of individuals.

$r(t)$ and the wage rate $w(t)$ are as follows:

$$r(t) = r(k(t)) = f\,'(k(t)) - \delta\,, \tag{4a}$$

$$w(t) = w(k(t)) = f\,(k(t)) - k(t)\,f\,'(k(t))\,, \tag{4b}$$

where δ is a constant capital depreciation rate.

2-2. Preferences, budget constraints and optimality conditions

The preference of type i individuals is expressed as follows:

$$\int_0^\infty \exp(-\rho t)\frac{c_i(t)^{1-1/\varepsilon_i}-1}{1-1/\varepsilon_i}dt \quad \text{for } i = P, R \quad \text{when } \varepsilon_i \neq 1\,, \tag{5a}$$

$$\int_0^\infty \exp(-\rho t)\log c_i(t)dt \quad \text{for } i = P, R \quad \text{when } \varepsilon_i = 1\,, \tag{5b}$$

where $\rho > 0$ stands for the rate of time preference, and $\varepsilon_i > 0$ for the intertemporal elasticity of substitution (IES). Assuming that each household supplies one unit of labor in each period, the budget constraint of type i individuals is given by

$$\dot{k}_i(t) = r(t)k_i(t) + w(t) - c_i(t) \quad \text{with } k_i(0) = \overline{k_i} \text{ for } i = P, R\,, \tag{6}$$

where $k_i(t)$ is type i individuals' capital stock, and $\overline{k_i}$ is the initial level. The associated Euler equations on consumption are

$$\dot{c}_i(t) = \varepsilon_i(r(t)-\rho)c_i(t) = \varepsilon_i(f\,'(k(t)) - \delta - \rho)c_i(t) \text{ for } i = P, R\,, \tag{7}$$

and the transversality conditions are

$$\lim_{t\to\infty} e^{-\rho t}c_i(t)^{-1/\varepsilon_i}k_i(t) = 0 \text{ for } i = P, R\,. \tag{8}$$

2-3. Aggregate behavior of the economy

Since, as (1) shows, λ_i is a population weight of type i individuals, we have

$$k(t) = \lambda_P k_P(t) + \lambda_R k_R(t)\,, \quad c(t) = \lambda_P c_P(t) + \lambda_R c_R(t)\,, \tag{9}$$

where $c(t)$ represents the per-capita consumption. Taking (9) into account, the following law of motion of the per-capita capital can be obtained from (4a), (4b) and (6):

$$\begin{aligned}\dot{k}(t) &= \lambda_P\dot{k}_P(t) + \lambda_R\dot{k}_R(t)\\ &= r(t)[\lambda_P k_P(t) + \lambda_R k_R(t)] + w(t)[\lambda_P + \lambda_R] - [\lambda_P c_P(t) + \lambda_R c_R(t)]\\ &= r(t)k(t) + w(t) - c(t) = f\,(k(t)) - \delta k(t) - c(t)\,. \end{aligned} \tag{10}$$

The above shows the resource constraint of the economy, i.e., the goods market equilibrium. Also, the following law of motion of the aggregate consumption can be obtained from (7):

$$\dot{c}(t) = \lambda_P \dot{c}_P(t) + \lambda_R \dot{c}_R(t) = (r(k(t)) - \rho)[\lambda_P \varepsilon_P c_P(t) + \lambda_R \varepsilon_R c_R(t)]. \tag{11}$$

If IES are the same across individuals, i.e., $\varepsilon_P = \varepsilon_R = \varepsilon$, then the above equation becomes

$$\dot{c}(t) = \varepsilon(r(k(t)) - \rho)c(t). \tag{12}$$

The aggregate behavior of the economy is characterized by (10), (12), and the initial and transversality conditions. In other words, as Caselli and Ventura (2000), among others, show, it becomes the same as in the standard Ramsey model with homogeneous individuals although consumption and capital can differ across individuals.

It is evident from (11) that the per-capita capital in the steady state is the same as the modified golden rule, k_{MGR}, which is determined by

$$f'(k_{MGR}) - \delta = r^* = \rho. \tag{13}$$

Once k_{MGR} is given, the resource constraint (10) determines the per-capita consumption in the stationary state c_{MGR}, which is given by

$$c_{MGR} = f(k_{MGR}) - \delta k_{MGR}. \tag{14}$$

Proposition 1

Regardless of individual heterogeneity in the intertemporal elasticity of substitution, the per-capita capital and consumption in the steady state are the same as those in the standard Ramsey model with homogeneous individuals.

2-4. Wealth distribution in the steady state

Since $\dot{k}_i(t) = 0$ in the steady state,

$$c_i^* = r^* k_i^* + w^* = \rho k_i^* + w^*, \tag{15}$$

where a variable with an asterisk denotes its steady state value. Since $f(k) = (r + \delta)k + w$ holds at any point in time, and hence $f(k_{MGR}) = (r^* + \delta)k_{MGR} + w^*$, (14) can be rewritten as follows:

$$c_{MGR} = (r^* + \delta)k_{MGR} + w^* - \delta k_{MGR} = r^* k_{MGR} + w^*,$$

and therefore

$$c_{MGR} = \rho k_{MGR} + w^* \text{ or } w^* = c_{MGR} - \rho k_{MGR} . \tag{16}$$

Substituting the above into (15),

$$c_i^* = r^* k_{MGR} + (c_{MGR} - \rho k_{MGR}) = \rho(k_i^* - k_{MGR}) + c_{MGR} \text{ for } i = P, R . \tag{17}$$

Since, as (13) shows, the slope of $c = f(k) - \delta k$ at (k_{MGR}, c_{MGR}) is ρ, (17) implies that all the points denoted by (k_i^*, c_i^*) are on the tangent line to the curve of $c = f(k) - \delta k$ at (k_{MGR}, c_{MGR}), i.e., (k_i^*, c_i^*) is on the line $S_P S_R$ in Fig. 1. If $k_i^* = 0$, then (17) becomes $c_i^* = -\rho k_{MGR} + c_{MGR}$. Substituting (16) into this equation gives $c_i^* = w^*$, which shows that consumption of an individual with no wealth in the steady state is equal to the wage. Thus, the long-run steady state equilibrium is represented by a straight line in the (k, c) plane in the Ramsey model with heterogeneity in IES, while it is shown as a single point in the (k, c) plane , that is (k_{MGR}, c_{MGR}), in the standard Ramsey model with no heterogeneity.

[Fig. 1 is around here.]

Proposition 2[5]

Suppose that the time-preference rate is the same across individuals, while the intertemporal elasticities of substitution and/or the initial wealth are different. Then, the long-run equilibrium is represented by a tangent line to the curve of $c = f(k) - \delta k$ *at* (k_{MGR}, c_{MGR}) *in the* (k, c) *plane.*

To derive the transition path to the long-run steady state, i.e., the dynamic general equilibrium path, in the standard Ramsey model we use the backward induction method. Without characterizing the transition path, we can determine the long-run steady state, which is shown by a single point in the (k, c) plane, such as (k_{MGR}, c_{MGR}) in Fig. 1. Since the initial condition is given only by the per-capita capital $k(0) = \bar{k}$, we can find the unique per-capita consumption in the initial period, $c(0)$, by using the backward induction method. The transition is represented by a unique path that connects the stationary state represented by (k_{MGR}, c_{MGR}) with the initial state represented by $(k(0), c(0))$.

[5] The same proposition is put forward in a more general setting in Nakamura (2020).

In contrast, the long-run steady state equilibrium in our model is represented by a straight line, such as $S_P S_R$ in Fig.1. Although the location and slope of the line are independent of the initial condition and heterogeneity in IES, the length depends crucially on them. Without deriving the transition path, the long-run steady state equilibrium cannot be fully characterized. In other words, the long-run equilibrium and the transition are simultaneously and interdependently determined. However, assuming a steady state instead of determining it prior to deriving the transition path allows us to use the backward induction method to derive the transition path. The next section applies this strategy to the simulation analysis in order to analyze changes in the wealth distribution over time and the effect of heterogeneity in IES on the dynamics of wealth distribution.

3. The simulation analysis

This section conducts simulations to examine how the aggregate and individual wealth evolve as the economy grows. In the simulations, we will use the following Cobb-Douglass production function:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}, \text{ i.e., } y_t = Ak_t^{\alpha} \text{ with } \alpha = 1/3.$$

Also, we assume $\rho = 0.05$ and $\delta = 0.1$.

Taking advantage of the results in the previous section, we fix a straight line in the (k, c) plane that shows the long-run steady state equilibrium. In all of the following simulations, type R individuals are assumed to have 25 percent more capital than the average in the steady state, while type P individuals have 25 percent less capital. In other words, the type R are richer than the type P by two thirds in terms of wealth. We also assume that the population of type R individuals is equal to that of type P individuals, i.e., $\lambda_R = \lambda_P = 0.5$. Hence, the wealth Gini coefficient in the steady state is 0.25 for all of the following simulations.

3-1. With no heterogeneity in IES

In this subsection we examine how the wealth distribution evolves during the transition in the Ramsey model in which individuals are homogeneous in preference, i.e., they have the same

IES in addition to the common rate of time preference. In concrete, we assume that $\varepsilon_R = \varepsilon_p = 1$, Panel (A) in Fig. 2 shows the phase diagram, in which the pink dashed line shows the equilibrium path of type R individuals, while the green dashed line shows the equilibrium path of type P individuals. The aggregate behavior is represented by the blue solid line. If no initial difference in wealth exists across individuals, our model becomes identical to the representative-agent Ramsey model. The blue line can therefore be interpreted as the equilibrium path in the standard Ramsey model. While Panel (B) shows the time paths of capital for the two types, Panel (C) shows the time path of the wealth Gini. [6]

[Fig. 2 is around here.]

The growth of the economy means increases in wealth of both types of individuals. Since type R's wealth increases faster than type P's, the wealth gap increases through time. At the same time, however, the total wealth also increases. As a result, the share of type R's in total wealth decreases, while that of type P's increases.[7] In other words, the growth in the economy as a whole expands the wealth gap between the two types but reduces the wealth Gini. One may suspect that this overall growth effect to reduce the wealth Gini depends crucially on the speed of growth, i.e., IES. However, this reduction effect is valid for the relevant rage of IES in our simple setting. For example, Fig. 3 shows the simulation results when $\varepsilon_R = \varepsilon_p = 0.5$. Although the transition in Fig. 3 is longer compared to Fig. 2 because the growth is lower, the basic properties of the dynamics are the same between Fig. 2 and Fig. 3. These observations lead us to the following remark.

[Fig. 3 is around here.]

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[6] See the appendix for the outline of computation code for the simulation analysis.

[7] For example, type R's wealth is 10 and type P's is 5 in the initial period, while type R's wealth is 20 and type P's is 12 in the stationary state. The wealth gap is 5 and the wealth share of type R's is 2/3, i.e., about 0.67 in the initial period, while the wealth gap is 8 and the wealth share of type R's is 5/8, i.e. 0.625.

Remark 1

Suppose that the initial wealth differs across individuals in the Ramsey model with no heterogeneity in the intertemporal elasticity of substitution. Then, the wealth gap expands as the economy grows, while the wealth Gini decreases over time.

3-2. Effects of heterogeneity in IES (1): Expanding inequality

Let us examine the effects of heterogeneity in IES. Since we have fixed the wealth distribution in the steady state, at least two distinct transitions must be considered. One is with expanding inequality, and the other is with shrinking inequality.

Even if the initial wealth is the same across individuals, the wealth can differ in the steady state if IES are different. Individuals with high IES start with low consumption and accumulate wealth rapidly. As a result, they have large wealth and consumption in the steady state. Fig. 4 shows the simulation results when $\varepsilon_R = 1$ and $\varepsilon_P = 0.5$. As Panel (A) shows, type R individuals start with low consumption and end up with large capital and consumption in the steady state, while type P individuals start with high consumption and end up with small capital and consumption.

[Fig. 4 is around here]

Remark 2

Suppose that the initial wealth is the same across individuals in the Ramsey model with heterogeneity in the intertemporal elasticity of substitution (IES). Then, the individuals with higher IES accumulate wealth rapidly and have a larger amount of wealth in the steady state. As a result, the wealth Gini increases as the economy grows.

Many empirical studies such as Attanasio et al. (2002), Blundell et al. (2014) and Browning (1995) have found that IES increases with wealth. Taking the fact into account that wealth is accumulated net savings, the empirical finding can be interpreted conversely, i.e., individuals with high IES accumulate wealth more rapidly than individuals with low IES. As a result, those with high IES have larger wealth than others. The above remark suggests it.

3-3. Effects of heterogeneity in IES (2): Shrinking inequality

Even if some individuals initially have a larger amount of wealth compared to others, the relative size is getting smaller in the transition if their IES are lower than the others. As explained in the previous subsection, the individuals with low IES start with high consumption and hence accumulate capital slowly. Fig. 5 shows the simulation results when $\varepsilon_R = 0.5$ and $\varepsilon_P = 1$. As Panel (A) shows, type R individuals accumulate wealth slowly by starting with high consumption, while type P individuals accumulate wealth rapidly by starting with low consumption. As a result, the wealth gap is smaller in the steady state than in the initial period.

[Fig. 5 is around here]

Although the graphs in Fig. 5 look similar to those in Fig. 2 because the wealth Gini starts at 0.5 and ends at 0.25 in the both figures, we should notice the difference in transition period. It takes about 30 periods for the Gini to reach 0.25 in Fig. 2, while it takes about 40 periods in Fig. 4. Of course, this observation is consistent with our intuition. Since individuals with high IES accumulate capital faster than those with low IES, the wealth gap shrinks more slowly when the rich have lower IES ($\varepsilon_R = 0.5$) than the poor ($\varepsilon_P = 1$) compared to when both types of individuals have the same high IES ($\varepsilon_R = \varepsilon_P = 1$).

3-4. U-shaped relationship with shoe on the other foot

If type R individuals, who have by assumption a larger amount of wealth than type P individuals in the steady state, have a smaller amount of wealth in the initial period, then the type R must overtake the type P in terms of wealth during the transition. The previous results have shown the possibility. If the type R have higher IES and start with smaller capital, then they can overtake the type P in terms of both capital and consumption before the economy reaches the stationary state. Fig. 6 shows the simulation results when $\varepsilon_R = 1$, $\varepsilon_P = 0.5$, and type R individuals have a smaller amount of capital in the initial period. The results can be summarized as the following remark.

[Fig. 6 is around here.]

Remark 3 (Inverted Kuznets Curve)

Suppose that individuals with high IES have small amounts of initial wealth, while individuals with low IES's have large amounts of initial wealth. Then, the possibility exists that the rich individuals in the initial period become relatively poor in the long run.

The above remark is important not only in the sense that it presents an interesting transition of the wealth Gini such as the U-shaped, i.e. an inverted Kuznets curve, but also in the sense that it demonstrates the importance of transition period in determining the dynamics of wealth distribution and the distribution in the steady state. The initial condition is crucial when the transition period is short, while the preference plays a dominant role when the period is long. As time passes, the effect of the initial condition gradually declines. Instead, that of the preference gradually increases. As a result, the U-shaped relationship between inequality and income can appear over time.

3-5. An Extension: Stone-Geary preferences

The above simulation method is not restricted to Ramsey models with constant IES. As an example, let us examine the model with the following Stone-Geary utility function:

$$\int_0^\infty \exp(-\rho t)\frac{(c_i(t)-\bar{c})^{1-(1/\varepsilon_i)}-1}{1-(1/\varepsilon_i)}dt \text{ for } i=P,R, \tag{18}$$

where $\bar{c}$ stands for minimum consumption. As Alvarez-Pelaez and Diaz (2005) and Obiols-Homs and Urrutia (2005) correctly point out, IES change as capital accumulates even if ε_i's are constant through time. Denoting the instantaneous utility function by $u_i(c_i(t))$, by definition, IES is expressed by

$$IES_i(t) = -\frac{\partial(\dot{c}_i(t)/c_i(t))}{\partial(\dot{u}_i{}'(c_i(t))/u_i{}'(c_i(t))} = -\frac{\partial(\dot{c}_i(t)/c_i(t))}{\partial(u_i{}''(c_i(t))\dot{c}_i(t)/u_i{}'(c_i(t)))} \text{ for } i=P,R.$$

With the specification in (18),

$$IES_i(t) = \frac{\varepsilon_i(c_i(t)-\bar{c})}{c_i(t)} = \varepsilon_i\left(1-\frac{\bar{c}}{c_i(t)}\right). \tag{19}$$

The above expression leads to the following remark.

Remark 4

Suppose that individual preferences is give by a Stone-Geary utility function. Then, the intertemporal elasticity of substitution becomes larger as consumption becomes larger, i.e., as wealth accumulates.

Now the law of motion of $k(t)$ and $c(t)$ is given by

$$\dot{k}(t) = \lambda_P \dot{k}_P(t) + \lambda_R \dot{k}_R(t) = f(k(t)) - \delta k(t) - c(t), \tag{20a}$$

$$\dot{c}(t) = \lambda_P \dot{c}_P(t) + \lambda_R \dot{c}_R(t) = (r(k(t)) - \rho)[\lambda_P \varepsilon_P (c_P(t) - \overline{c}) + \lambda_R \varepsilon_R (c_R(t) - \overline{c})]. \tag{20b}$$

While (20a) is the same as (10), (20b) corresponds to (11). Although (20b) is different from (11), the both give the same modified golden rule level of capital accumulation in the steady state, k_{MGR}. Hence Propositions 1 and 2 remain true, and consequently we can apply the same method as before to the model with Stone-Geary preferences.

Suppose, for example, $\varepsilon_R = \varepsilon_p = 1$ and $\overline{c} = 0.5$. If $c_R(0) = 1$ and $c_P(0) = 0.8$, then, it is calculated from (19) that $IES_R(0) = 0.5$ and $IES_p(0) = 0.375$. Thus, Stone-Geary preferences lowers IES, with lowering the poor's IES more than the rich's. Since individuals with high IES accumulate wealth rapidly, Stone-Geary preferences have an effect of expanding the wealth gap.

[Fig. 7 is around here.]

Fig. 7 shows the simulation results when $\varepsilon_R = \varepsilon_P = 1$ and $\overline{c} = 0.5$. In the initial period, the both types have a small amount of wealth although type R have slightly more than type P. Since type R therefore have higher consumption than type P, type R's IES is higher than type P's. As a result, type R accumulate wealth more rapidly than type P, as Panel (B) shows. In other words, since the expanding effect by the Stone-Geary preference is dominant during the early stage of transition, the wealth Gini increases, as Panel (C) shows. However, as Panel (A) shows, the gap in consumption between the two types shrinks as time goes. Since the gap in IES also shrinks, the behavior of the model becomes similar to that of the Ramsey model with no heterogeneity in IES. We know from Remark 1 that the Gini decreases during the late stage of transition, as is shown in Panel (C). Consequently, the inverted U-shaped relationship between inequality and income is observed.

Remark 5 (Kuznets Curve)

Introducing Stone-Geary preferences into a standard Ramsey model with no heterogeneity in the intertemporal elasticity of substitution, the inverted U-shaped relationship between income and inequality, i.e., the Kuznets Curve, can emerge.

4. Concluding Remarks

Both the initial conditions and individual preferences play a crucial role in the determination of the dynamics of wealth distribution and the distribution in the steady state. Although most previous studies have paid much attention to time-preference rather than intertemporal substitution, as this paper shows, the latter also plays an important role. To carefully examine the wealth distribution in both the long run and short run, we must pay enough attention to individual preferences, characterized by time preference and intertemporal substitution, as well as to the initial conditions. In general, as the transition becomes longer, the effect of initial condition becomes smaller, while that of individual preferences becomes larger. Our simulation results have clearly shown that the roles of the intertemporal elasticity of substitution (IES) in the transitional dynamics and steady state of wealth distribution cannot be overlooked.

To clarify the role of heterogeneity in IES, we have only conducted the simulation analysis. We believe that the backward induction method used in this paper can also be applied to the calibration. It is hoped that our attempt stimulates future research.

Appendix: Pseudo Code

The appendix gives the outline of computation code for the simulation, i.e., the pseudo code, in this paper.

procedure Calculation of Transition Dynamics

Setting the parameters

Calculate the steady state

$k_{MGR} = \left(\dfrac{\alpha A}{\delta + \rho} \right)^{-\frac{1}{1-\alpha}}$

$c_{MGR} = Ak_{MGR}{}^{\alpha} - \delta k_{MGR}$

$k_{MGR}^{R} = 1.25 k_{MGR}$, $k_{MGR}^{P} = 0.75 k_{MGR}$

$c_{MGR}^{R} = 1.25 c_{MGR}$, $c_{MGR}^{P} = 0.75 c_{MGR}$

Calculation of Coefficient for Reverse Shooting Method

$dw = 0.5 A\alpha(\alpha - 1) k_{MGR}^{\alpha-2}$

$dr = 0.5 A\alpha(1-\alpha) k_{MGR}^{\alpha-1}$

$dr = dr + dw + \rho$

for $t = 0 : dt : T$ do

$i = i + 1$

$c'_R = c_{RT}$, $c'_P = c_{PT}$

$k'_R = k_{RT}$, $k'_R = k_{PT}$

Calculation of $\dot{c}_{Rt}$ and $\dot{c}_{Pt}$: Reverse Shooting Method (Judd(1998))

for $j = 1 : 4$ do

if $T = 0$ then

$\dot{c}_{Rt} = -0.5 * \left(dss^2 + \sqrt{dss^2 + 2\varepsilon_R (c_{RMGR} - \overline{c})(1-\alpha)\alpha k_{MGR}^{\alpha-2}} \right)$

$\dot{c}_{Pt} = -0.5 * \left(dss^2 + \sqrt{dss^2 + 2\varepsilon_P (c_{PMGR} - \overline{c})(1-\alpha)\alpha k_{MGR}^{\alpha-2}} \right)$

else $T \neq 0$ then

$\dot{c}_{Rt} = -\varepsilon_{Rt}(c_{Rt} - \overline{c})(\alpha k_t^{\alpha-1} - \rho - \delta)$

$\dot{c}_{Pt} = -\varepsilon_{Pt}(c_{Pt} - \overline{c})(\alpha k_t^{\alpha-1} - \rho - \delta)$

end if

4th-order Runge-Kutta Methods

$dc_{Rj} = \dot{c}_{Rt} dt$

$c'_R = c_{Rt} + 0.5\dot{c}_{Rj}$

if $j = 3$ then

$c'_R = c_R + \dot{c}_{Rj}$

end if

$dc_{Pj} = \dot{c}_{Pt} dt$

$c'_P = c_{Pt} + 0.5\dot{c}_{Pj}$

if $j = 3$ then

$c'_P = c_P + \dot{c}_{Pj}$

end if

end for

Calculation of Consumption

$c_{R+1} = c_{Rt} + \frac{1}{6}(dc_{R1} + 2dc_{R2} + 2dc_{R3} + dc_{R4})$

$c_{P+1} = c_{Pt} + \frac{1}{6}(dc_{P1} + 2dc_{P2} + 2dc_{P3} + dc_{P4})$

$c_{t+1} = 0.5c_{Rt+1} + 0.5c_{Pt+1}$

Calculation of $\dot{k}_{it}$

for $j = 1:4$ do

$\dot{k}_{Rt} = -(rk_{Rt} + w - c_{Rt})$

$\dot{k}_{Pt} = -(rk_{Pt} + w - c_{Pt})$

4th-order Runge-Kutta Methods

$dk_{Rj} = \dot{k}_{Rt} dt$

$k'_R = k_R + 0.5\dot{k}_{Rj}$

if $j = 3$ then
$k'_R = k_R + \dot{k}_{Rj}$
end if

$dk_{Pj} = \dot{k}_{Pt} dt$
$k'_P = k_P + 0.5\dot{k}_{Rj}$
if $j = 3$ then
$k'_R = k_P + \dot{k}_{Pj}$
end if
end for

Calculation of Capital
$k_{R+1} = k_{Rt} + \frac{1}{6}(dk_{R1} + 2dk_{R2} + 2dk_{R3} + dk_{R4})$
$k_{P+1} = k_{Pt} + \frac{1}{6}(dk_{P1} + 2dk_{P2} + 2dk_{P3} + dk_{P4})$
$k_{t+1} = 0.5k_{Rt+1} + 0.5k_{Pt+1}$

Calculation of Prices
$r_{t+1} = \alpha A k_{t+1}^{\alpha-1} - \delta$
$w_{t+1} = (1-\alpha) A k_{t+1}^{\alpha}$
end for
end procedure

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Figures

Fig. 1

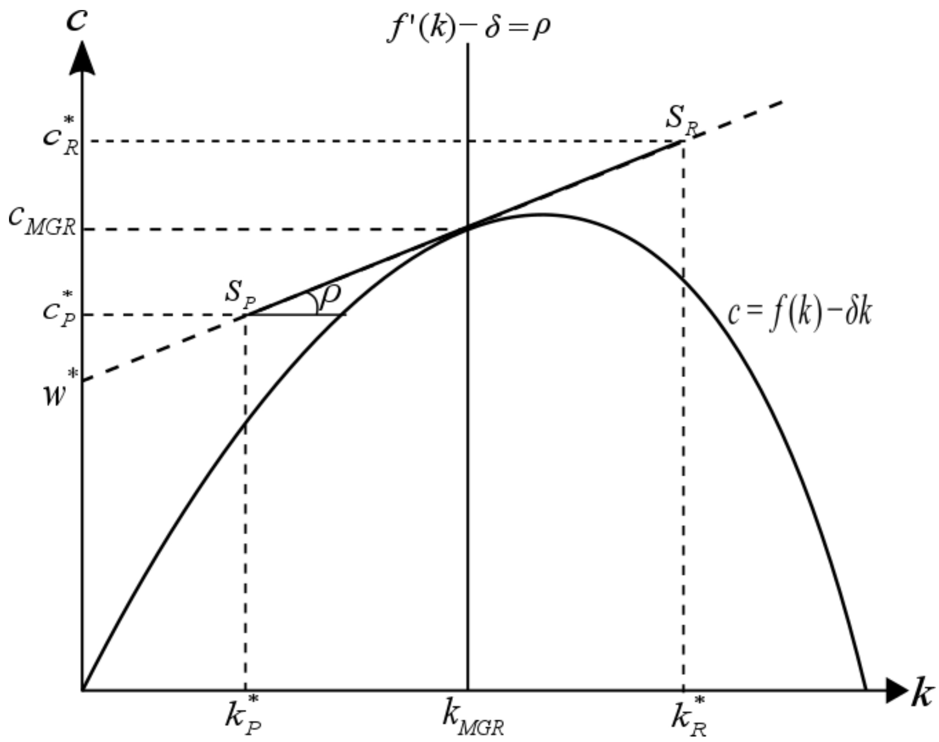
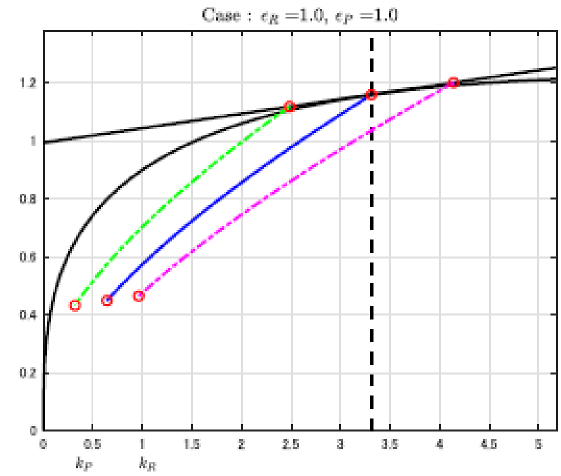
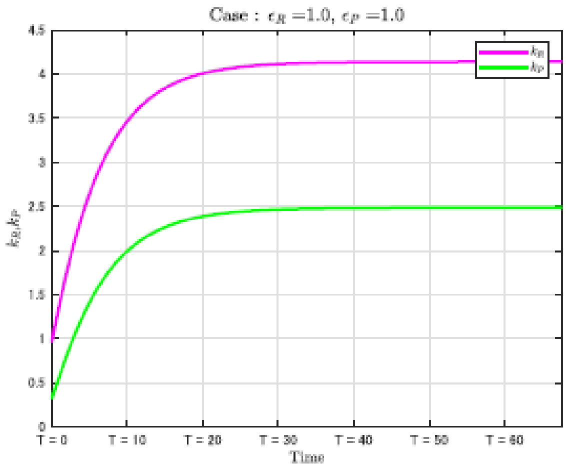


Fig. 1 Steady State Distribution

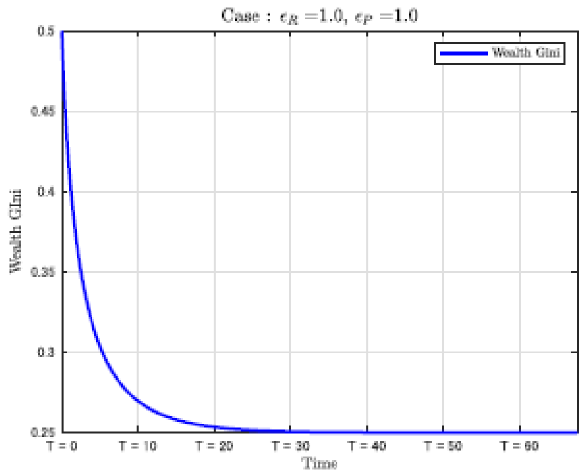
Fig. 2



(A) Phase Diagram



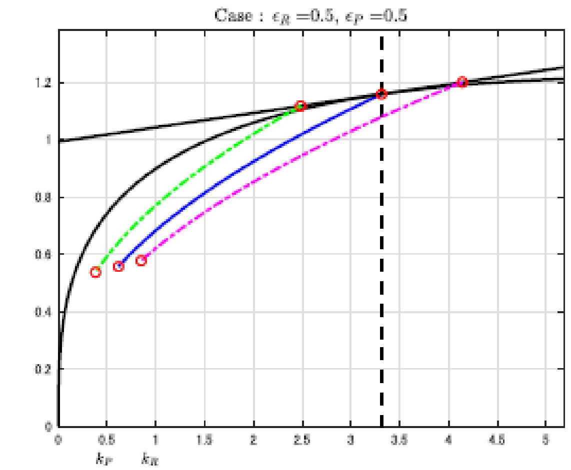
(B) Time Paths of Capital



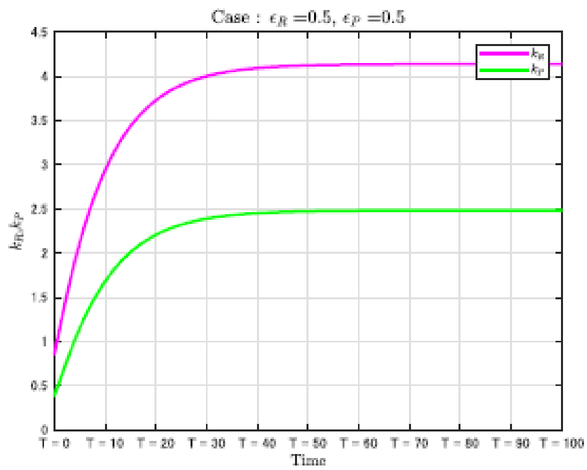
(C) Time Path of Wealth Gini

Fig. 2 Standard Ramsey Model When $\varepsilon_R = \varepsilon_P = 1$

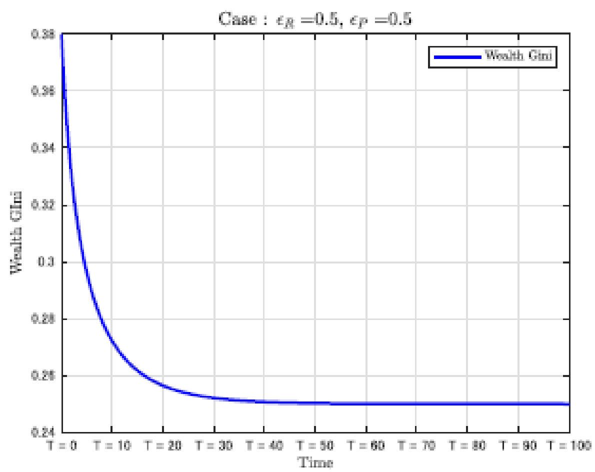
Fig. 3



(A) Phase Diagram



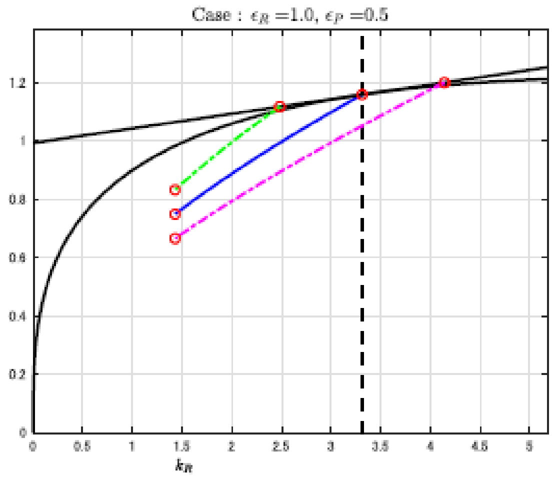
(B) Time Paths of Capital



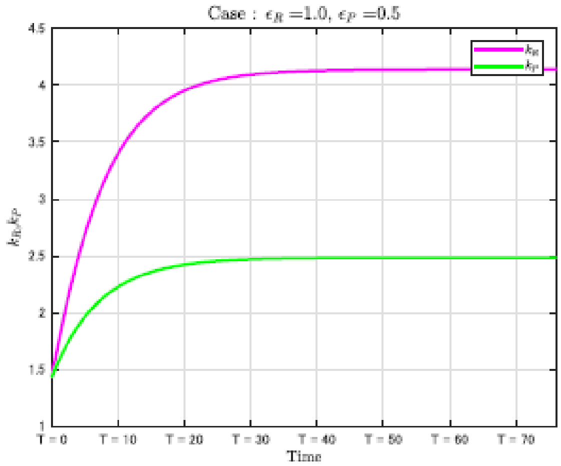
(C) Time Path of Wealth Gini

Fig. 3 Standard Ramsey Model When $\varepsilon_R = \varepsilon_P = 0.5$

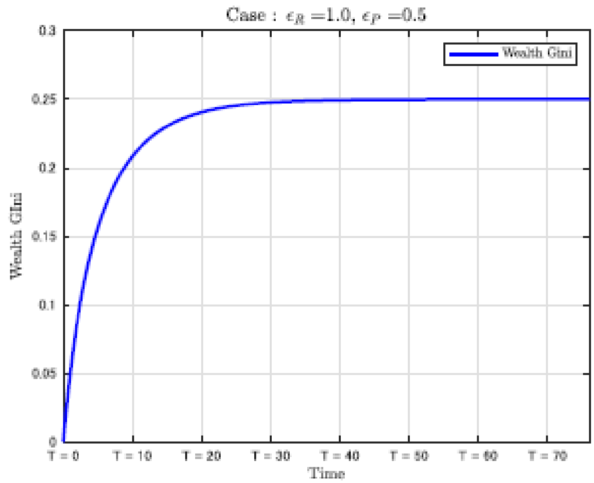
Fig. 4



(A) Phase Diagram



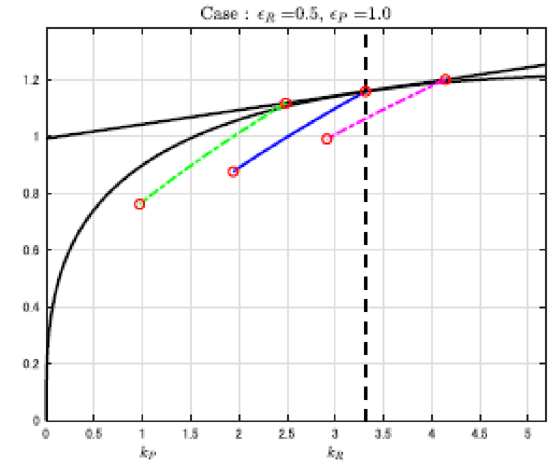
(B) Time Paths of Capital



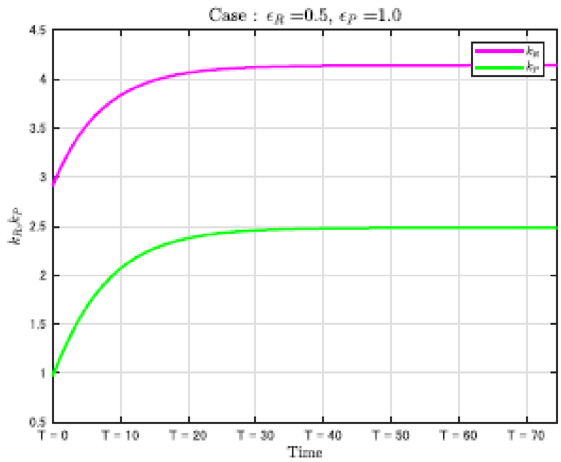
(C) Time Path of Wealth Gini

Fig. 4 Effects of Heterogeneity in IES (1) ($\varepsilon_R = 1$ and $\varepsilon_P = 0.5$)

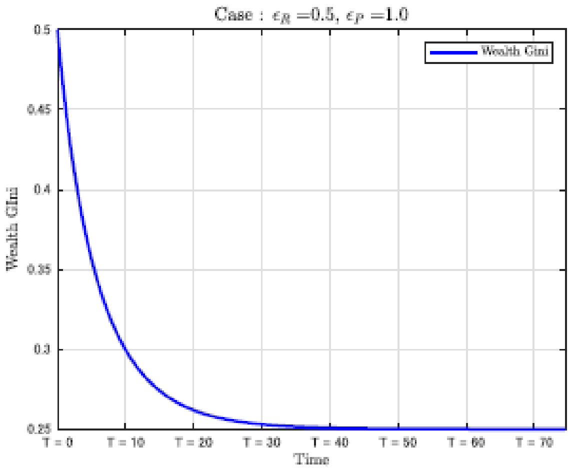
Fig. 5



(A) Phase Diagram



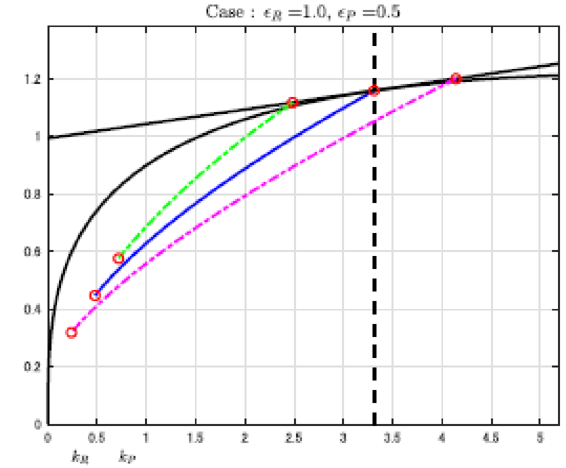
(B) Time Paths of Capital



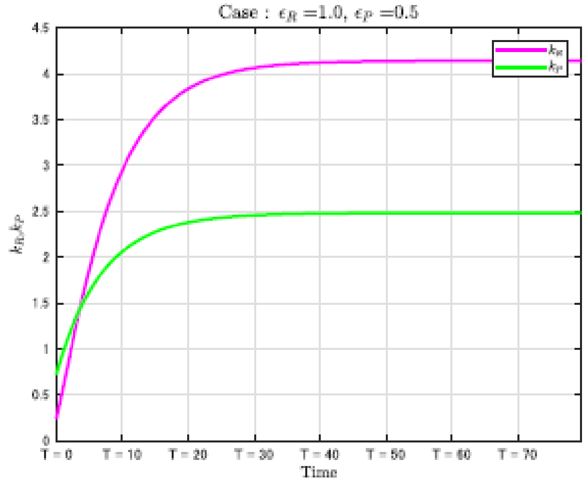
(C) Time Path of Wealth Gini

Fig. 5 Effects of Heterogeneity in IES (2) ($\varepsilon_R = 0.5$ and $\varepsilon_P = 1$)

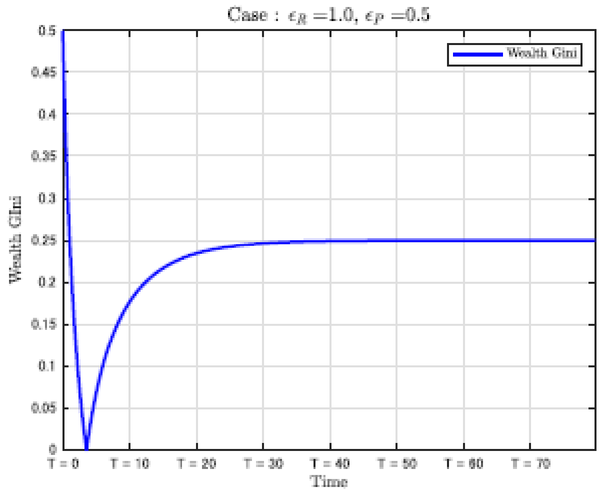
Fig. 6



(A) Phase Diagram



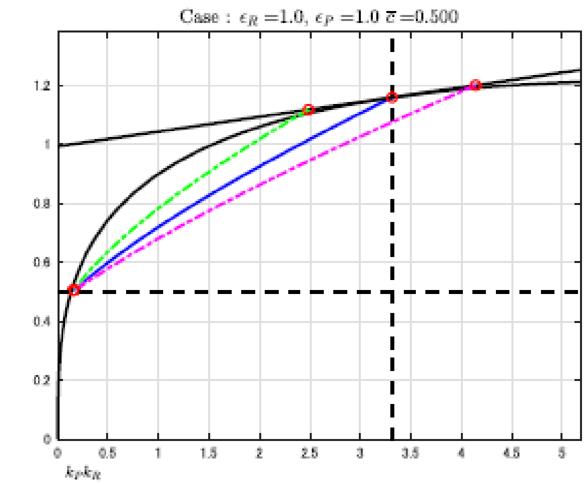
(B) Time Paths of Capital



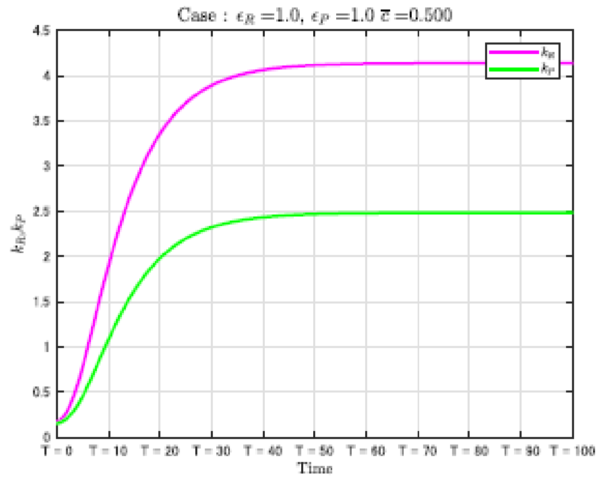
(C) Time Path of Wealth Gini

Fig. 6 U-shaped Relationship between Income and Inequality

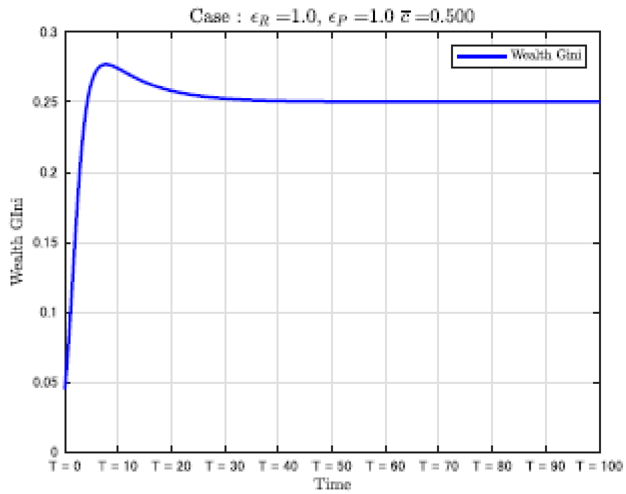
Fig. 7



(A) Phase Diagram



(B) Time Paths of Capital



(C) Time Path of Wealth Gini

Fig. 7 Inverted U-shaped Relationship under Stone-Geary Preferences