An introduction of a simple monetary policy with savings taxation in the overlapping generations model

Taro Ikeda

March 2018

Discussion Paper No.1810

GRADUATE SCHOOL OF ECONOMICS

KOBE UNIVERSITY

ROKKO, KOBE, JAPAN

An introduction of a simple monetary policy with savings taxation in the overlapping generations model

Taro Ikeda*
Graduate School of Economics, Kobe University

March 6, 2018

Abstract

In this paper, we introduce a simple monetary policy with savings taxation into Samuelson's (1958) overlapping generations model. In our model, we confirm that the real market interest rate increases in response to an increase in the rate of the savings taxation as a policy lending rate.

Keywords: overlapping generations model; monetary policy; savings taxation

JEL classifications: E10; E20; E52

1. Introduction

In this paper, we introduce a simple monetary policy with the taxation of savings into Samuelson's (1958) overlapping generations model.[1] We assume that only the central bank provides the means for individuals of current generations to save: the central bank imposes a savings tax rate (lending rate) τ on one unit of individual savings demand, and provides $(1-\tau)$ units of savings to the individuals of the current generation. In turn, the improvement of the central bank's balance sheet that results from this lending provides the same amount of real money supply for the young people of the next generations.[2]

This transfer of a real quantity of money increases the endowments of the individuals. Therefore, the real money transfer may improve the utility of individuals by increasing their consumption feasibility. In this sense, our model answers the question "Why do individuals hold money?" with the response "Because it is obligatorily given by the policy" In our model, money does not yield direct utility for individuals, but it is held by agents as an additional endowment as a result of the monetary policy requirements.

This paper mainly focuses on the fundamental equilibrium condition defined by Samuelson (1958). The fundamental equilibrium condition implies that savings are cleared, as was excess demand for goods, such that they equal zero at the equilibrium. In particular, we consider how the equilibrium market interest rate responds to the change of the lending rate set by the savings taxation policy.[3] In other words, we are interested in the manner in which the market interest rate safeguards the equilibrium against the policy interventions.

---

* Research Fellow, Graduate School of Economics, Kobe University, Japan (Email: taroikd@people.kobe-u.ac.jp).

[1] We restrict the discussion to the case of agents who live two periods. In addition, most of the variables considered are measured in real terms.

[2] Throughout the paper, we assume that $\tau < 1$. This implies that the central bank cannot require more goods from the individual than the quantity that he/she originally provided to the bank

[3] As purpose of this paper is to provide the most basic model based on Samuelson (1958), we do not discuss welfare issues as well as production problems and others.

This simple model suggests that the market interest rate moves in the same direction as the policy lending rate under the fundamental equilibrium conditions.

2. Introduction of the monetary policy

Suppose that the young people of generation t cannot make savings—that is, make consumption-loans to themselves in old age—without transforming a portion $s_t(t)$ of their t-period goods into savings through central banking, where the only central bank is the sole provider of savings.

However, transforming goods into savings imposes a tax of rate τ on the young: after receiving lending revenue $\tau s_t(t)$ from the young, the central bank gives them an amount of savings equal to $(1-\tau)s_t(t)$. Then, in turn, the young people of generation t lend their savings $(1-\tau)s_t(t)$ to themselves when they are old, based on the market interest rate $r(t)$. Finally, the young people obtain their savings with an after-tax-interest rate, $(1-\tau)r(t)s_t(t)$, in old age.[4]

The lending revenue of the central bank increases the assets side of the central bank balance sheet. Then, we assume that the central bank issues new real money to provide for the young people of generation $t+1$. The new real money, which is the same amount as the lending revenue, increases the liabilities side of the sheet, and balances the account. As a result, our monetary policy satisfies:

(1) $$m_{t+1}(t+1) = \tau s_t(t).$$

By the balance sheet equation (1), an increase of the lending interest rate τ leads to an increase of the real money transfer to the young of the next generation, $m_{t+1}(t+1)$. The real money transfer does not require significant time lags because the taxation of savings in period t is implemented at the end of the period t.

3. Individual optimization

In the purest Marshallian case, the individual of generation t has a log-utility function expressed as $u_t\big(c_t(t), c_t(t+1)\big) = \log c_t(t) + \log c_t(t+1)$ (Samuelson, 1958). Then, the maximization problem of the agent becomes as follows:

(2) $$\sum_{i=t}^{t+1} \log c_t(i) \quad \text{subject to:}$$

$$c_t(t) + \frac{c_t(t+1)}{(1-\tau)r(t)} = [\omega_t(t) + m_t(t)] + \frac{\omega_t(t+1)}{(1-\tau)r(t)},$$

where $c_t(t)$ and $c_t(t+1)$ denote the consumption amounts of generation t for youth t and old age $t+1$, respectively. In addition, $\omega_t(t)$ and $\omega_t(t+1)$ denote the endowments of generation t for youth and old age, respectively. Note the fact that, in the lifetime budget constraint, the interest rate imposed on savings $s_t(t)$ corresponds to the after-tax-interest rate $(1-\tau)r(t)$ and the real money transfer $m_t(t)$ is added to the endowment $\omega_t(t)$ in the agent's youth.

Then, the Euler equation of this problem becomes:

$$\frac{\partial}{\partial c_t(t)} u_t \Big/ \frac{\partial}{\partial c_t(t+1)} u_t = (1-\tau)r(t),$$

---

[4] If there are no special conditions and the market interest rate $r(t)$ is constant, an increase of lending rate τ in period t decreases the after-tax-interest rate $(1-\tau)r(t)$ in period $t+1$. Therefore, the well-known positive relationship between the interest rate and the money supply is superficially preserved in this model.

and, therefore, we obtain the following:

(3) $$c_t(t) = \frac{1}{(1-\tau)r(t)} c_t(t+1).$$

By equation (3) and the lifetime budget constraint, the optimal savings functions for the individual's youth and old age, respectively, can be derived as follows:

(4) $$s_t(t) \equiv [\omega_t(t) + m_t(t)] - c_t(t) = \frac{1}{2}[\omega_t(t) + m_t(t)] - \frac{\omega_t(t+1)}{2(1-\tau)r(t)},$$

(5) $$s_t(t+1) \equiv \omega_t(t+1) - c_t(t+1) = \frac{1}{2}\omega_t(t+1) - \frac{(1-\tau)r(t)}{2}[\omega_t(t) + m_t(t)].$$

4. Fundamental equilibrium conditions

This section provides the fundamental equilibrium conditions defined by Samuelson (1958).

For any $t \geq 1$, there exists $N(t)$ young of generation t and $N(t-1)$ old of generation $t-1$. Then, the aggregate savings of each period yield the following fundamental equilibrium condition:

(6) $$0 = N(t)s_t(t) + N(t-1)s_{t-1}(t),$$

for all $t \geq 1$.

This condition requires good market clearing, in the sense that the savings play the role of excess demand for goods. We also assume that the population is finite and constant over time (i.e., $N(t) = N(t-1) = N$) to avoid difficulties for the aggregations within the periods.

5. Equilibrium

The population of the economy begins in period 0. In period 0, young people of generation 0 are born, and they are the only people who exist in this period. Then, in period 1, the young of generation 1 are born, and they coexist with the old of generation 0 (who were young in period 0).[5] Thereafter, the age distribution of the young and old is reproduced forever.

Here, we calculate the fundamental equilibrium condition (6) with $N(t) = N(t-1) = N$:

$$0 = s_t(t) + s_{t-1}(t),$$

for each time period. Note that the real money supply $m_0(0)$ is always zero because there were no people saving before period 0.

Below, we present the fundamental equilibrium conditions in each period.

Period 0 (in which $\tau s_{-1}(-1) = m_0(0) = 0$):

$$s_0(0) + s_{-1}(0) = \left\{\frac{1}{2}[\omega_0(0) + m_0(0)] - \frac{\omega_0(1)}{2(1-\tau)r(0)}\right\} + 0 = 0,$$

$$s_0(0) + s_{-1}(0) = \left\{\frac{1}{2}[\omega_0(0)] - \frac{\omega_0(1)}{2(1-\tau)r(0)}\right\} + 0 = 0,$$

$$\omega_0(0) = \frac{\omega_0(1)}{(1-\tau)r(0)}.$$

Period 1 (in which $\tau s_{-1}(-1) = m_0(0) = 0$ and $s_0(0) = 0$ by $\omega_0(0)$):

[5] The population of generation 0 disappears at the end of period 1.

$$s_1(1)+s_0(1)=\left\{\frac{1}{2}[\omega_1(1)+m_1(1)]-\frac{\omega_1(2)}{2(1-\tau)r(1)}\right\}+\left\{\frac{1}{2}\omega_0(1)-\frac{(1-\tau)r(0)}{2}[\omega_0(0)+m_0(0)]\right\}=0,$$

$$s_1(1)+s_0(1)=\left\{\omega_1(1)-\frac{\omega_1(2)}{(1-\tau)r(1)}\right\}+\{0\}=0,$$

$$\omega_1(1)=\frac{\omega_1(2)}{(1-\tau)r(1)}.$$

Period 2 (preconditions are abbreviated):

$$s_2(2)+s_1(2)=\left\{\frac{1}{2}[\omega_2(2)+m_2(2)]-\frac{\omega_2(3)}{2(1-\tau)r(2)}\right\}+\left\{\frac{1}{2}\omega_1(2)-\frac{(1-\tau)r(1)}{2}[\omega_1(1)+m_1(1)]\right\}=0,$$

$$s_2(2)+s_1(2)=\left\{[\omega_2(2)]-\frac{\omega_2(3)}{(1-\tau)r(2)}\right\}+\{0\}=0,$$

$$\omega_2(2)=\frac{\omega_2(3)}{(1-\tau)r(2)}.$$

⋮

By these conditions, we can deduce the fundamental equilibrium condition for period t as follows:

(7) $$\omega_t(t)=\frac{\omega_t(t+1)}{(1-\tau)r(t)}.$$

Therefore, the real market interest rate $r(t)$ satisfies the following relationship:

(8) $$(1-\tau)r(t)=\frac{\omega_t(t+1)}{\omega_t(t)},$$

under the fundamental equilibrium condition. Accordingly, the equilibrium market interest rate $r(t)$ offsets the change of τ to retain a fraction of the individual's lifetime endowments at each time period.[6]

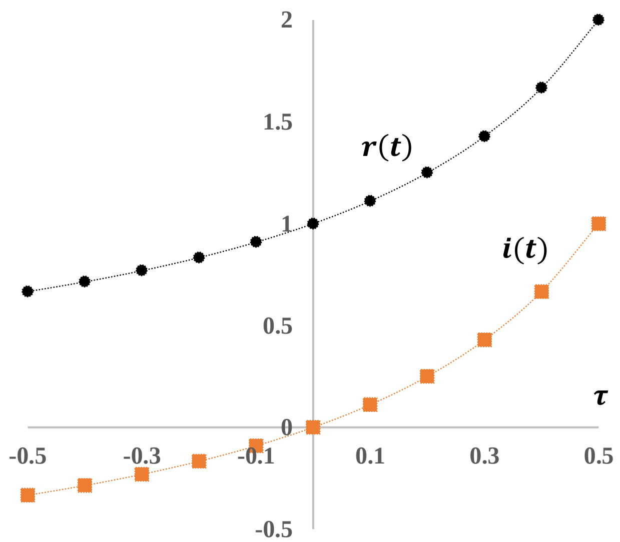

Figure 1. Relationship between lending interest rate τ and market interest rate $r(t)$ in the fundamental equilibrium with lifetime endowment $(\omega_t(t),\omega_t(t+1))=(1,1)$

[6] The equilibrium market interest rate (7) clears the two sets of savings in each period $(s_t(t),s_{t-1}(t))$ separately. Therefore, considering money in this equilibrium is meaningless.

Figure 1 represents the response of the real gross market interest $r(t)$ to the savings tax rate τ in the fundamental equilibrium, given the lifetime endowment $\left(\boldsymbol{\omega_t(t)}, \boldsymbol{\omega_t(t+1)}\right) = (\mathbf{1}, \mathbf{1})$. In this specified economy, the market interest rate rises in response to the increase of the policy lending rate (the rate of savings taxation). Further, in this zero-growth equilibrium, a negative rate of savings taxation implies a negative net market interest rate, $i(t) \equiv r(t) - 1$.

6. Constant growth within generation and equilibrium market interest rate

So far, we have investigated the proportional relationship between the market interest rate and the rate of savings taxation in the equilibrium. However, we have not yet determined the equilibrium market interest rate itself. Therefore, let us consider the fundamental equilibrium with a constant economic growth rate *within generation*, $g \equiv \frac{\omega_t(t+1)-\omega_t(t)}{\omega_t(t)} \in (-1,1)$ (see Figure 2 for example).

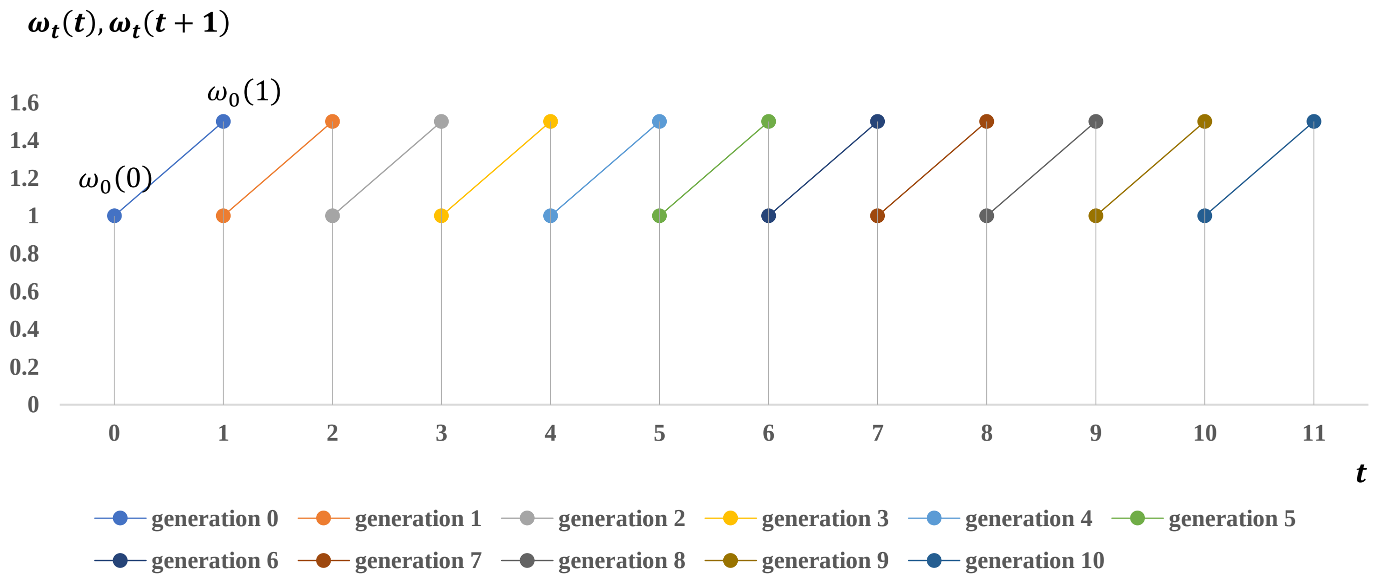

Figure 2. Constant growth rate within generation with $\boldsymbol{g = 0.5}$ and $\boldsymbol{\omega_t(t) = 1}$.

From the assumption, it follows that:

(9) $$1 + g = \frac{\omega_t(t+1)}{\omega_t(t)}.$$

Inserting (9) into (8), the equilibrium real market interest rate satisfies the following:

(10) $$r = \frac{1+g}{1-\tau}.$$

In the fundamental equilibrium with a constant growth rate g, the market interest rate moves to keep the after-tax-interest rate at $(1+g)$ in response to the change of the savings taxation rate. How? For $\tau < 1$, the market interest rate satisfies:

(11) $$\frac{\partial}{\partial \tau} r = (1+g)\left(\frac{1}{1-\tau}\right)^2 > 0.$$

Therefore, the equilibrium market interest rate is increasing in the rate of the savings taxation.

In sum, we confirm that the market interest rate increases in response to the monetary policy interest rate increasing.

7. Conclusion

In this paper, we introduce a simple monetary policy involving savings taxation into Samuelson's (1958) overlapping generations model. We confirm that, in our model, the real market interest rate increases in response to a rise of the rate of the savings taxation as a policy lending rate.

Reference

Samuelson, Paul A. 1958. "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money." Journal of Political Economy, Vol. 66, No. 6, December, pp. 467–482.

Appendix for "An introduction of a simple monetary policy by saving taxations in the overlapping generations model"

March 6, 2018

A1. Calculations for "Individual optimization (Section 3)"

Let us consider the following individual maximization problem:

$$(1-a) \qquad \sum_{i=t}^{t+1} \log c_t(i) \quad \text{subject to}$$

$$c_t(t) + \frac{c_t(t+1)}{(1-\tau)r(t)} = [\omega_t(t) + m_t(t)] + \frac{\omega_t(t+1)}{(1-\tau)r(t)}.$$

The lifetime budget constraint in $(1-a)$ can be split into the budget constraints of young and old as follows:

$$c_t(t) + s_t(t) \leq \omega_t(t) + m_t(t),$$

$$c_t(t+1) \leq \omega_t(t+1) + (1-\tau)r(t)s_t(t).$$

At this point, note that we define the individual savings of each age as $s_t \equiv \omega_t(t) + m_t(t) - c_t(t)$, $s_t(t+1) \equiv r(t)s_t(t)$.

Then, we set the Lagrangian for the maximization problem as follows:

$$\mathcal{L} = \sum_{i=t}^{t+1} \log c_t(i) + \lambda \left\{ [\omega_t(t) + m_t(t)] + \frac{\omega_t(t+1)}{(1-\tau)r(t)} - \left[c_t(t) + \frac{c_t(t+1)}{(1-\tau)r(t)} \right] \right\}.$$

Then, we obtain the first-order conditions as follows:

$$\frac{\partial}{\partial c_t(t)}\mathcal{L} = \frac{1}{c_t(t)} - \lambda = 0,$$

$$\frac{\partial}{\partial c_t(t+1)}\mathcal{L} = \frac{1}{c_t(t+1)} - \lambda \frac{1}{(1-\tau)r(t)} = 0.$$

Accordingly,

$$\frac{1}{c_t(t+1)} = \frac{1}{c_t(t)} \frac{1}{(1-\tau)r(t)},$$

$$\frac{c_t(t+1)}{c_t(t)} = \frac{1}{c_t(t)} \Big/ \frac{1}{c_t(t+1)} = (1-\tau)r(t).$$

Therefore, we obtain

$$(2-a) \qquad c_t(t) = \frac{1}{(1-\tau)r(t)} c_t(t+1).$$

Inserting $(2-a)$ into the lifetime budget constraint, we obtain

$$c_t(t) + c_t(t) = [\omega_t(t) + m_t(t)] + \frac{\omega_t(t+1)}{(1-\tau)r(t)}.$$

Then, we drive the optimal consumption demands for the individual when young and old as follows:

$$c_t(t) = \frac{1}{2}[\omega_t(t) + m_t(t)] + \frac{\omega_t(t+1)}{2(1-\tau)r(t)},$$

And,

$$c_t(t+1) = \frac{1}{2}(1-\tau)r(t)[\omega_t(t) + m_t(t)] + \frac{1}{2}\omega_t(t+1).$$

In addition, we obtain the optimal saving functions for the individual when young and old as:

$$s_t(t) = [\omega_t(t) + m_t(t)] - c_t(t) = [\omega_t(t) + m_t(t)] - \left\{\frac{1}{2}[\omega_t(t) + m_t(t)] + \frac{\omega_t(t+1)}{2(1-\tau)r(t)}\right\},$$

$$s_t(t) = \frac{1}{2}[\omega_t(t) + m_t(t)] - \frac{\omega_t(t+1)}{2(1-\tau)r(t)},$$

And,

$$s_t(t+1) = \omega_t(t+1) - c_t(t+1) = \omega_t(t+1) - \left\{\frac{(1-\tau)r(t)}{2}[\omega_t(t) + m_t(t)] + \frac{1}{2}\omega_t(t+1)\right\},$$

$$s_t(t+1) = \frac{1}{2}\omega_t(t+1) - \frac{(1-\tau)r(t)}{2}[\omega_t(t) + m_t(t)].$$

A2. Calculations for "Equilibrium (Section 5)"

Here, we calculate the fundamental equilibrium condition (6) with $N(t) = N(t-1) = N$:

$$0 = s_t(t) + s_{t-1}(t),$$

by each period. Note that the real money supply $m_0(0)$ is always zero because we find no people with savings prior to period 0.

Below, we present the fundamental equilibrium conditions in each period.

Period 0 (with $\tau s_{-1}(-1) = m_0(0) = 0$):

$$s_0(0) + s_{-1}(0) = \left\{\frac{1}{2}[\omega_0(0) + m_0(0)] - \frac{\omega_0(1)}{2(1-\tau)r(0)}\right\} + 0 = 0,$$

$$s_0(0) + s_{-1}(0) = \left\{\frac{1}{2}[\omega_0(0)] - \frac{\omega_0(1)}{2(1-\tau)r(0)}\right\} + 0 = 0,$$

$$\omega_0(0) = \frac{\omega_0(1)}{(1-\tau)r(0)}.$$

Period 1 (with $\tau s_{-1}(-1) = m_0(0) = 0$ and $s_0(0) = 0$ by $\omega_0(0)$):

$$s_1(1) + s_0(1) = \left\{\frac{1}{2}[\omega_1(1) + m_1(1)] - \frac{\omega_1(2)}{2(1-\tau)r(1)}\right\} + \left\{\frac{1}{2}\omega_0(1) - \frac{(1-\tau)r(0)}{2}[\omega_0(0) + m_0(0)]\right\} = 0,$$

$$s_1(1) + s_0(1) = \left\{\frac{1}{2}[\omega_1(1)] - \frac{\omega_1(2)}{2(1-\tau)r(1)}\right\} + \left\{\frac{1}{2}\omega_0(1) - \frac{(1-\tau)r(0)}{2}\left[\frac{\omega_0(1)}{(1-\tau)r(0)}\right]\right\} = 0,$$

$$s_1(1) + s_0(1) = \left\{\omega_1(1) - \frac{\omega_1(2)}{(1-\tau)r(1)}\right\} + \{0\} = 0,$$

$$\omega_1(1) = \frac{\omega_1(2)}{(1-\tau)r(1)}.$$

Period 2 (preconditions are abbreviated):

$$s_2(2) + s_1(2) = \left\{\frac{1}{2}[\omega_2(2) + m_2(2)] - \frac{\omega_2(3)}{2(1-\tau)r(2)}\right\} + \left\{\frac{1}{2}\omega_1(2) - \frac{(1-\tau)r(1)}{2}[\omega_1(1) + m_1(1)]\right\} = 0,$$

$$s_2(2) + s_1(2) = \left\{\frac{1}{2}[\omega_2(2)] - \frac{\omega_2(3)}{2(1-\tau)r(2)}\right\} + \left\{\frac{1}{2}\omega_1(2) - \frac{(1-\tau)r(1)}{2}\left[\frac{\omega_1(2)}{(1-\tau)r(1)}\right]\right\} = 0,$$

$$s_2(2) + s_1(2) = \left\{[\omega_2(2)] - \frac{\omega_2(3)}{(1-\tau)r(2)}\right\} + \{0\} = 0,$$

$$\omega_2(2) = \frac{\omega_2(3)}{(1-\tau)r(2)}.$$

Using these conditions, we deduce the fundamental equilibrium condition for period t as follows:

$$(3-a) \qquad \omega_t(t) = \frac{\omega_t(t+1)}{(1-\tau)r(t)}.$$

A3. Calculations for "Constant growth within generation and equilibrium market interest rate (Section 6)"

In the main text, we obtain the equilibrium market interest rate as

$$(4-a) \qquad r(\tau) = \frac{1+g}{1-\tau}.$$

Here, we calculate the partial derivative of $(4-a)$ on the domain $\tau < 1$. Accordingly, the partial derivative of $r(\tau)$ with respect to τ becomes

$$(5-a) \qquad \frac{\partial}{\partial \tau} r(\tau) = (1+g)\left(\frac{1}{1-\tau}\right)^2 > 0.$$

Therefore, we confirm that the market interest rate r increasingly responds to the rate of saving taxation τ.