Optimal Funded Pension for Consumers with Heterogeneous Self-Control

Kazuki Kumashiro *
Graduate School of Economics, Kobe University

August 18, 2015

Abstract

We consider a public funded pension scheme. In standard models, it is well known that a funded pension does not improve social welfare since the consumers can maximize their lifetime utilities by private savings. In contrast, recent works show that if consumers suffer from their temptation on spending, the funded pension can have a beneficial effect on social welfare. This is because the funded pension has a property as a commitment device, which private savings does not have. However, though it is natural that the degree of temptation varies among individuals, this is not reflected in existing pension policies. We provide the pension scheme that regards this point and maximizes social welfare. A government, that does not know the degrees of the temptation of each consumer, proposes a list that consists of some pairs of pension premium and pension return to consumers, and each consumer chooses one of this before the decision of consumption. The problem is how to design this list. Under the assumption that a utility function for consumption c is $\log(c)$, we show that the optimal pension scheme does not transfer incomes among consumers. This result does not depend on the distribution of the degrees of temptation.

Key Words: Self-Control; Overconsumption; Funded Pension

1 Introduction

We consider a public funded pension scheme. Social security policies including a funded pension have some objectives. One of them is income redistribution. This is common objective in almost all social security policies such as funded pensions, social aid by taxation and pay-as-you-go pensions. Other important role of social securities is that as forced saving devices, which is specific to funded pensions. This is because funded pensions assure subscribers that they can receive retirement incomes at least their contribution. However, it is well known that for consumers in standard economic models, funded pensions do not improve social welfare (e.g. Samuelson, 1975). For example, suppose that a consumer found that his optimal amount of saving is \$1000 per month. When there is no pension policy, he privately saves \$1000. When a pension is introduced and the premium that he pays is \$500 per month, he pays it and privately saves \$500 because his optimal saving is \$1000. Since his total amount of saving does not change between before and after the introducing of the pension, so does his lifetime utility. Of course, if the premium is larger

*Corresponding address: Graduate School of Economics, Kobe University, 2-1 Rokkodai-cho, Nada-ku, Kobe, Hyogo, 657-8501, JAPAN. E-mail: 111e103e@stu.kobe-u.ac.jp

than \$1000, he goes into oversaving that decreases his lifetime utility. Thus the pension does not improve social welfare.

One of factors to justify funded pensions is temptation that leads consumers to overconsumption (Diamond, 1977). Even if a consumer knows how much to save his income, he often cannot help but overspending more than the amount when he has disposable income. Once he faces this temptation, he feels psychological cost to self-control his wasteful spending. So it is ideal for him not to face temptation by a valid way for commitment. This is why we need funded pensions as a forced saving policy.

However, in the context of temptation, most of pension schemes presently used are not suitable. It is natural that the degrees of temptation are not the same for all consumers. While there are consumers that spend much of their income as soon as they earns them and regret their myopic behavior, there are also consumers that can spend their income farsightedly. Since pension schemes affect the behavior of consumers, they should be designed with thinking of this point. However, in present pension schemes, a premium that each consumer pays is determined mainly by his income level, not by his degree of temptation. Our purpose is to design a pension scheme that maximizes social welfare considering the difference in the degree of temptation.

Some literature study policies as a way to resist temptation. It is shown that funded pensions improve social welfare (Gul and Pesendorfer, 2004). This is because, as stated above, they enable consumers to avoid temptation with less self-control cost. In addition, pay-as-you-go pension is also able to improve social welfare if temptation is sufficiently large (Kumru and Thanopoulos, 2008). Other way to relieve self-control cost is to make consumption relatively unattractive. This works because the cost arises from the gap between the attractiveness of normatively desirable alternatives and that of tempting alternative. Krusell et al. (2010) shows that subsidies for savings improve social welfare for this reason. Though these studies are important, their research objects are economies with consumers whose degrees of temptation are the same for all of them. On the other hand, our model can analyze the situation in which the degrees are different for different consumers.

Since we need a model that describes decisions with temptation and self-control, we apply the model proposed by Gul and Pesendorfer (2001), henceforth GP. They considered two kinds of preferences over alternatives, normative preferences and temptation preferences, which are represented by functions u and v, respectively. Using u and v, the valuation of menu (a set of alternatives) M is defined by

$$W(M) \equiv \max_{x \in M}[u(x) + v(x)] - \max_{y \in M} v(y). \tag{1}$$

GP assumes that, for a given menu M, a decision maker chooses an alternative $x \in M$ that maximizes $u(x) + v(x)$. The intuition is that he chooses a compromise between normative desirability and temptation desirability after facing temptation. Let $x^* \in M$ be an alternative that maximizes $u(x) + v(x)$. Then (1) is rewritten as

$$W(M) = u(x^*) - \left\{ \max_{y \in M} v(y) - v(x^*) \right\}.$$

The term $\max_{y \in M} v(y) - v(x^*)$ represents self-control cost. For instance, suppose that a consumer faces a menu $M = \{s, h\}$, where s and h represents salad and hamburger, respectively. And suppose that he wants to diet but he likes meat. In GP model, u corresponds to a desirability of health and v does to a desirability of having what he likes. Thus $u(s) > u(h)$ and $v(h) > u(s)$ holds. If $u(s) + v(s) > u(h) + v(h)$, he chooses salad for his lunch. Note that salad does not maximize v, thus he gives up satisfaction of temptation as much as $v(h) - v(s)$. GP defined self-control cost by this difference. Generalizing this concept, we have the expression above.

Menus correspond to budget sets in consumption choice problems. If a pension premium can shrink the budget sets, it works as a commitment device and decreases self-control cost. Formally, assume that a pension premium makes a menu M shrink to $M' \subset M$. Then we have $\max_{y \in M'} v(y) \leqslant \max_{y \in M} v(y)$, thus $\max_{y \in M'} v(y) - v(x^*) \leqslant \max_{y \in M} v(y) - v(x^*)$ follows. Especially, if the inequality holds strictly, the self-control cost decreases strictly.

As an alternative approach to analyze myopic behaviors, we may use (quasi) hyperbolic discounting model provided by Laibson (1997). Actually, some literature on economic policies employ this approach (e.g. Roeder, 2014). The model describes the situation in which preferences are time inconsistent. However, the model cannot describe self-control cost explicitly. Since we want to treat effects of temptation and self-control separately, we employ GP model that is proper for our objective.

Section 2 introduces notations and assumptions used in the study and looks at the consumption-saving decision of a consumer with a self-control preference. Section 3 studies identical-type and two-type economies as benchmarks and shows the monotonicity of an optimal pension. In section 4, we generalize the results in section 3 to finitely many types and continuous-types model. We show that the results are robust for generalization. In section 5, we discuss the effects of borrowing constraints and robustness for universal domains.

2 Model

The only difference between standard models of pension and our model is that consumers have preference with temptation and self-control. There are consumers and an government. At first, the government offers a set of pairs of pension payout and pension premium to each consumer before his consumption decision in working age. Then the consumers choose one of pairs from the set. After a payment of premium the consumer has chosen, he decides how much to consume in working age.

2.1 Budget Constraint

We standardize the population of consumer to be 1. Each consumer is endowed with an identical income of $I \in \mathbb{R}_{++}$. Let $R \in \mathbb{R}_+$ and $P \in [0, I]$ be a pension premium and a pension payout, respectively. We assume the upper bound of P to be I to rule out situations in which consumers borrow money to pay the premiums. We call a pair of R and P *pension plan*. For the simplicity of notation, we define the set of possible pension plans as $T \equiv \mathbb{R}_+ \times [0, 1]$. A *pension schedule* is a set of pension plans.

In period 0, each consumer chooses a pension plan $\tau \in T$, from a pension schedule $S \subseteq T$. At the same time, he has to pay the government the pension premium he has chosen. In period 1, he decides the amounts of consumption $c_1 \in \mathbb{R}_+$ and saving $I - P - c_1$ in working age. We assume that consumers can also borrow money in period 1 with putting up their pension income as collateral. Thus, if $I - P - c_1$ is strictly greater (less) than 0, it represents the amount of saving (borrowing). Let r, $\rho \in \mathbb{R}_{++}$ be the interest rate for saving and borrowing, respectively. Thus if a consumer chose $\tau = (R, P)$, he can borrow up to $\frac{R}{1+\rho}$. About interest rates, we assume that $\rho > r > 0$. This assumption is natural from a practical standpoint. In period 2, he receives a pension income R and decides consumption in old age $c_2 \in \mathbb{R}_+$.

The budget set for consumers having chosen $\tau = (R, P)$ is summarized as follows.

$$B(\tau) \equiv \left\{ (c_1, c_2) \in \mathbb{R}_+^2 \colon \begin{array}{c} c_1 \leqslant I - P + \frac{R}{1+\rho}, \\ I - P - c_1 \geqslant 0 \Rightarrow c_2 \leqslant (1+r)(I - P - c_1) + R, \\ I - P - c_1 < 0 \Rightarrow c_2 \leqslant (1+\rho)(I - P - c_1) + R \end{array} \right\}.$$

Denote the set of all possible $B(\tau)$ by $\mathscr{B}$.

2.2 Preference

Consumers have self-control preference introduced by GP. We begin with defining two kinds of utility functions representing preference over consumption. One is normative utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ and the other is temptation utility function $v\colon \mathbb{R}_+ \to \mathbb{R}$. To express the temptation, v is specialized as $v(\cdot,\cdot) \equiv \lambda u(\cdot,\cdot)$, where $\lambda \in \Lambda \subseteq \mathbb{R}_+$ denotes the strength of the temptation. This form of temptation utility is also used in Gul and Pesendorfer (2004). We assume the value λ differs across agent and is private information. For any $\lambda \in \Lambda$, let n_λ be a proportion of consumer, that is, n_λ satisfies $0 \leqslant n_\lambda \leqslant 1$ and $\sum_{\lambda\in\Lambda} n_\lambda = 1$. In section 4.2, we consider continuous Λ. Then the proportion is represented by a distribution function $F\colon \Lambda \to [0,1]$ with density function $f\colon \Lambda \to \mathbb{R}_+$.

We impose a few assumptions about the normative utility function.

Assumption 2.1. *u is twice continuously differentiable.*

Assumption 2.2. *u is strictly concave and satisfies $u'(c) > 0$ and $u''(c) < 0$.*

Assumption 2.3. $\lim_{c\to 0} u'(c) = \infty$.

For example, these assumption is satisfied if normative utility functions is $u(c) = \log(c)$ or $u(c) = c^\alpha$ with $\alpha \in (0,1)$. In the latter half of this paper, we assume $u(c) \equiv \log(c)$ to derive sharper results. Applying the utility function by Gul and Pesendorfer (2001), the preference over budget sets is represented by the function $W\colon \mathscr{B} \times \mathbb{R}_+ \to \mathbb{R}$ satisfies that

$$\begin{aligned} W(B(\tau);\lambda) &= \max_{(c_1,c_2)\in B(\tau)} \big[u(c_1) + \delta u(c_2) + \lambda u(c_1)\big] - \max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1) \\ &= \max_{(c_1,c_2)\in B(\tau)} \big[(1+\lambda)u(c_1) + \delta u(c_2)\big] - \max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1), \end{aligned}$$

where $\delta \in (0,1)$ is a discount factor.

Using assumption 2.2, we have the following claim.

Claim 2.1. $\max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1) = \lambda u\left(I - P + \frac{R}{1+\rho}\right)$.

This is obvious by $u'(c) > 0$ (assumption 2.2) and the constraint for c_1 is $c_1 \leqslant I - P + \frac{R}{1+\rho}$.

2.3 Consumption

In this section, we consider the consumption problem contained in $W(B(\tau);\lambda)$, that is, $\max_{(c_1,c_2)\in B(\tau)} \big[(1+\lambda)u(c_1)+\delta u(c_2)\big]$. When the consumer of type $\lambda \in \Lambda$ chooses a pension plan $\tau \in T$, let his optimal consumption in period t be $c_t^\lambda(\tau)$.

There can be the three types of consumption: positive saving, zero saving (or balanced) and negative saving (or borrowing). The type of optimal consumption depends on the pension plan that a consumer has chosen. Intuitively, since a large premium brings too little disposable income at a working age, it may be necessary for him to borrow to achieve his desired level of consumption. Formally, we separate T as

$$
\begin{aligned}
T_s^\lambda &\equiv \{\tau \in T \colon c_1^\lambda(\tau) < I - P\} \\
T_b^\lambda &\equiv \{\tau \in T \colon c_1^\lambda(\tau) = I - P\} \\
T_d^\lambda &\equiv \{\tau \in T \colon c_1^\lambda(\tau) > I - P\}.
\end{aligned}
$$

τ_s^λ, τ_b^λ and τ_d^λ denote the typical elements of T_s^λ, T_b^λ and T_d^λ, respectively. λ in superscript is omitted if there is no threat of confusion. Note that if $\tau \in T_b$, $c_2^\lambda(\tau) = R$, since $u'(c) > 0$ and income in period 2 is only pension income R.

We have the necessary conditions for optimality in different form for τ_s and τ_d since the budget constraint in period 2 differs. In this consumption optimization problem, an objective function is $U(c_1, c_2) \equiv (1 + \lambda)u(c_1) + \delta u(c_2)$. So (the absolute value of) the marginal rate of substitution of c_1 for c_2 is

$$
\frac{\frac{\partial U(c_1,c_2)}{\partial c_1}}{\frac{\partial U(c_1,c_2)}{\partial c_2}} = \frac{(1+\lambda)u'(c_1)}{\delta u'(c_2)}.
$$

If $\tau \in T_s$, the budget line has the slope of $1 + r$. So the necessary condition for τ_s is [1]

$$
\begin{aligned}
&\frac{(1+\lambda)u'(c_1(\tau_s))}{\delta u'(c_2(\tau_s))} = 1 + r \\
\Leftrightarrow &(1+\lambda)u'(c_1(\tau_s)) = (1+r)\delta u'(c_2(\tau_s)). \qquad (2)
\end{aligned}
$$

On the other hand, if $\tau \in T_d$, the budget line has a slope of $1 + \rho$. So the necessary condition for τ_d is

$$
\begin{aligned}
&\frac{(1+\lambda)u'(c_1(\tau_d))}{\delta u'(c_2(\tau_d))} = 1 + \rho \\
\Leftrightarrow &(1+\lambda)u'(c_1(\tau_d)) = (1+\rho)\delta u'(c_2(\tau_d)). \qquad (3)
\end{aligned}
$$

Since a budget line kinks at the consumption vector for τ_b, the slope of the line there is not defined. Thus we cannot derive a condition as in other cases. However, we have at least the following inequality:

$$
\begin{aligned}
&1 + r < \frac{(1+\lambda)u'(c_1(\tau_b))}{\delta u'(c_2(\tau_b))} < 1 + \rho \\
\Leftrightarrow &(1+r)\delta u'(c_2(\tau_b)) < (1+\lambda)u'(c_1(\tau_b)) < (1+\rho)\delta u'(c_2(\tau_b)).
\end{aligned}
$$

2.4 Government

The government offers consumers a pension schedule $S \equiv \{\tau_\lambda \in T : \lambda \in \Lambda\} \subset T$, a set of pension plans. The objective of the government is to maximize the (expected) aggregated welfare of consumers while controlling S to satisfy the following conditions:

$$
W(B(\tau_\lambda); \lambda) \geqslant W(B(\tau'); \lambda), \ \forall \lambda \in \Lambda, \ \forall \tau' \in S \qquad \text{(IC)}
$$

$$
(1+r)\sum_{\lambda \in \Lambda} n_\lambda P_\lambda \geqslant \sum_{\lambda \in \Lambda} n_\lambda R_\lambda. \qquad \text{(FB)}
$$

[1] There is not any corner solution because of assumption 2.3.

The first condition states that the consumers of type $\lambda \in \Lambda$ will choose a plan $\tau(\lambda)$ at their own initiative; in other words, this is the condition for not giving consumers any incentive to report their types untruthfully. Note that this is not necessary when types are public information since then government can force any plan on consumers.

The second condition states that S is feasible. We assume that pension management interest rate r is the same as that of the private saving. This means that there is no difference between the government and private banks in the ability to manage assets.

We also assume that we can ignore an individual rationality condition since the pension is managed by the government, which has the power of consumers' participation.

3 Benchmark

3.1 Identical-type consumers

In this chapter, we consider the simple case in which all consumers have identical type, that is, $\Lambda = \{\lambda\}$ and this is common knowledge. Then the condition (IC) can be ignored. Moreover, since λ is identical, the condition (FB) is rewritten as private condition, that is, $(1+r)P(\lambda) \geqslant R(\lambda)$.

Consider the individual welfare function $W(\tau;\lambda)$. Define $\sigma_\lambda \colon T \to \mathbb{R}$ to be the marginal rate of substitution of R for P on $\tau \in T$, that is,

$$\sigma_\lambda(\tau) = -\frac{\frac{\partial W(\tau;\lambda)}{\partial R}}{\frac{\partial W(\tau;\lambda)}{\partial P}}.$$

Since we cannot represent the optimal consumption in a general form, we have to consider the optimal pension plan separately. However, the following lemma makes it easy to analyze.

Lemma 3.1. *For any individual welfare level, an indifference curve corresponding to the level in R-P plane satisfies following properties.*

(i) $\sigma_\lambda(\tau_s)$ *is greater than or equal to* $\frac{1}{1+r}$*, where equality holds if and only if* $\lambda = 0$*.*

(ii) $\sigma_\lambda(\tau_d)$ *is equal to* $\frac{1}{1+\rho} < \frac{1}{1+r}$*.*

(iii) Every indifference curve is differentiable.

Proof of property (i). Consider arbitrary $\tau \in T_s$. Then we have

$$W(B(\tau);\lambda) = (1+\lambda)u(c_1(\tau)) + \delta u\big(c_2(\tau)\big) - \lambda u\left(I - P + \frac{R}{1+\rho}\right).$$

The marginal welfare of R is calculated as follows:

$$\begin{aligned}\frac{\partial W(B(\tau);\lambda)}{\partial R} =& (1+\lambda)u'(c_1(\tau))\frac{\partial c_1(\tau)}{\partial R} + \delta u'\big(c_2(\tau)\big)\big)\frac{\partial c_2(\tau)}{\partial R} \\ &- \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)\end{aligned}$$

Since $c_2(\tau_s) = (1+r)(I - P - c_1(\tau_s)) + R$, $\frac{\partial c_2(\tau_s)}{\partial R} = -(1+r)\frac{\partial c_1(\tau_s)}{\partial R} + 1$. Hence we have

$$\begin{aligned}\frac{\partial W(B(\tau);\lambda)}{\partial R} =& \Big\{(1+\lambda)u'(c_1(\tau)) - (1+r)\delta u'(c_2(\tau))\Big\}\frac{\partial c_1(\tau)}{\partial R} \\ &+ \delta u'(c_2(\tau)) - \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right) \\ =& \delta u'(c_2(\tau)) - \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right).\end{aligned}$$

The second equality follows by the first order condition for optimal consumption, that is, $(1+\lambda)u'(c_1(\tau)) = (1+r)\delta u'(c_2(\tau))$. Moreover, the last line can be rewritten as

$$\frac{1+\lambda}{1+r}u'(c_1(\tau)) - \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right),$$

and this is strictly positive. To see this, use assumption 2.2. Since $c_1(\tau) \leqslant I - P + \frac{R}{1+\rho}$ and $u''(c) < 0$, it holds that $u'(c_1(\tau)) \geqslant u'\left(I - P + \frac{R}{1+\rho}\right)$. By $\rho > r$ and $\lambda \geqslant 0$, we have $\frac{\partial W(\tau;\lambda)}{\partial R} > 0$.

Next, the marginal welfare of P is

$$\begin{aligned}\frac{\partial W(B(\tau);\lambda)}{\partial P} =& (1+\lambda)u'(c_1(\tau))\frac{\partial c_1(\tau)}{\partial P} - \delta u'(c_2(\tau))(1+r)\left(1 + \frac{\partial c_1(\tau)}{\partial P}\right) \\ &+ \lambda u'\left(I - P + \frac{R}{1+\rho}\right) \\ =& \Big\{(1+\lambda)u'(c_1(\tau)) - (1+r)\delta u'(c_2(\tau))\Big\}\frac{\partial c_1(\tau)}{\partial P} \\ &- (1+r)\delta u'(c_2(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right) \\ =& -(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right).\end{aligned}$$

By the same ways as in the previous paragraph, the equalities are follows and the last line is strictly negative.

Thus, $\sigma_\lambda(\tau)$ for $\tau \in T_s$ is

$$\sigma_\lambda(\tau) = -\frac{\frac{\partial W(B(\tau);\lambda)}{\partial R}}{\frac{\partial W(B(\tau);\lambda)}{\partial P}} = \frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}.$$

We show that this is greater than or equal to $\frac{1}{1+r}$. Since $\rho > r$, we have

$$\begin{aligned}&-(1+\lambda)u'(c_1(\tau)) + \frac{1+r}{1+\rho}\lambda u'\left(I - P + \frac{R}{1+\rho}\right) \\ &\qquad \leqslant -(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right),\end{aligned}$$

where equality holds if $\lambda = 0$. Furthermore, since $u'\left(I - P + \frac{R}{1+\rho}\right)$ it holds only if $\lambda = 0$. Using the first order condition,

$$\begin{aligned}&-(1+r)\delta u'(c_2(\tau)) + \frac{1+r}{1+\rho}\lambda u'\left(I - P + \frac{R}{1+\rho}\right) \\ &\qquad \leqslant -(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right).\end{aligned}$$

Note that the right-hand side is strictly negative. Thus this inequality is rewritten as,

$$\begin{aligned}
&\frac{-\delta u'(c_1(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \geqslant \frac{1}{1+r} \\
\Leftrightarrow &\sigma_\lambda(\tau) \geqslant \frac{1}{1+r}.
\end{aligned}$$

□

Proof of Property (ii). Calculating $\sigma_\lambda(\tau)$ for $\tau \in T_d$, we have

$$\begin{aligned}
\sigma_\lambda(\tau) =& \frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \\
=& \frac{-(1+\rho)\delta u'(c_2(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\rho)\left\{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)\right\}}.
\end{aligned}$$

By the first order condition for $\tau \in T_d$ in the consumption problem, we have

$$(1+\rho)\delta u'(c_2(\tau)) = (1+\lambda)u'(c_1(\tau)).$$

Therefore $\sigma_\lambda(\tau) = \frac{1}{1+\rho} < \frac{1}{1+r}$. □

Proof of Property (iii). It is enough to show that there exist $\tau \in T_s$ and $\tau \in T_d$ such that $\sigma_\lambda(\tau)$ is equal to that for $\tau \in T_b$, respectively. We have found that the slopes are

$$\frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}, \text{ if } \tau \in T_s \tag{4}$$

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}, \text{ if } \tau \in T_b \tag{5}$$

$$\frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}, \text{ if } \tau \in T_d. \tag{6}$$

Fix arbitrary $\hat{P} \in [0, I]$ and choose any pair $(R, \hat{P})$ such that $(R, \hat{P}) \in T_s$. Similarly, choose any pair $(R', \hat{P})$ such that $(R', \hat{P}) \in T_b$. Denote $\sup_{(R,\hat{P})\in T_s} R$ by $\overline{R}(\hat{P})$. Note that consumption function is continuous. According to the definitions of T_s and T_b, it follows that

$$\begin{aligned}
&c_1(R, \hat{P}) \to I - \hat{P},\ c_2(R, \hat{P}) \to \overline{R}(\hat{P}) \\
&c_1(R'\hat{P}) \to I - \hat{P},\ c_2(R', \hat{P}) \to \overline{R}(\hat{P}) \\
&\text{as } R \to \overline{R}(\hat{P}) \text{ and } R' \to \overline{R}(\hat{P}).
\end{aligned}$$

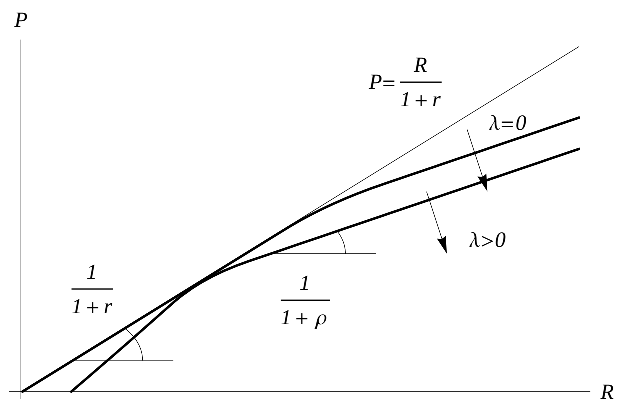

Figure 1: Indifference curves

Then we have

$$\lim_{R\to\overline{R}(\hat{P})}\sigma_\lambda(R,\hat{P}) = \frac{-\delta u'(\overline{R}(\hat{P})) + \frac{\lambda}{1+\rho}u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)}{-(1+\lambda)u'(I-\hat{P})+\lambda u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)}$$

$$\lim_{R'\to\overline{R}(\hat{P})}\sigma_\lambda(R',\hat{P}) = \frac{-\delta u'(\overline{R}(\hat{P})) + \frac{\lambda}{1+\rho}u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)}{-(1+\lambda)u'(I-\hat{P})+\lambda u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)},$$

that is, $\lim_{R\to\overline{R}(\hat{P})}\sigma_\lambda(R,\hat{P}) = \lim_{R'\to\overline{R}(\hat{P})}\sigma_\lambda(R',\hat{P})$. Therefore the indifference curve is smoothly continuous at $(\overline{R}(\hat{P}),\hat{P})$. Similarly we can prove the latter half of property 3. □

Even though the consumption patterns are different for τ, this lemma states that indifference curves for fixed welfare are smooth. Thus we can use the standard method of welfare maximization. The indifference curve is shown in Figure 1. Here, the pension plan that satisfies feasibility is in the upper-left area of the line $P = \frac{R}{1+r}$. Hence we find that optimal plans are determined at the tangent point of the indifference curve and the line of $P = \frac{R}{1+r}$. According to the lemma 3.1, the indifference curve does not have any tangent to the line $P = \frac{R}{1+r}$ at $\tau \in T_s$ and $\tau \in T_d$ if $\lambda > 0$. Thus, if τ is optimal, it is included in T_b except the case of $\lambda = 0$. If $\lambda = 0$, since $\sigma_\lambda(\tau_s) = \frac{1}{1+r}$, all $\tau_s \in T_s$ are optimal.

Let us consider optimal τ. First, we assume that $\lambda > 0$. As we discussed, it is enough to obtain optimal plans to consider $\tau \in T_b$. The optimal plan is determined at the tangent point of the indifference curve and $P = \frac{R}{1+r}$. Here, we have

$$\sigma_\lambda(\tau_b) = \frac{-\delta u'(R) + \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P)+\lambda u'\left(I-P+\frac{R}{1+\rho}\right)}.$$

Thus the necessary condition is

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P)+\lambda u'\left(I-P+\frac{R}{1+\rho}\right)} = \frac{1}{1+r}.$$

This can be rearranged as

$$\lambda = \frac{(1+r)\delta u'(R) - u'(I-P)}{u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}.$$

The denominator on the right-hand side is obviously positive since $I - P \leqslant I - \frac{\rho-r}{1+\rho}P$ and $\frac{\rho-r}{1+\rho} < 1$. However, we cannot state whether the numerator is positive or negative. Indeed, $R = (1+r)P$ in the solution then the numerator is $(1+r)\delta u'((1+r)P) - u'(I-P)$. This is positive for a sufficiently small P but negative for a sufficiently large P. Since λ is greater than 0, in order to characterize the optimal plan, we consider only P such that $(1+r)\delta u'((1+r)P) - u'(I-P) > 0$. Define $Q\colon \mathscr{P} \to \Lambda$ as

$$Q(P) \equiv \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P\right)}, \tag{7}$$

where $\mathscr{P} \equiv \{P \in [0, I]\colon (1+r)\delta u'((1+r)P) - u'(I-P) > 0\}$.

Let us see what properties Q has. Importantly, the following lemma says that the optimal τ is unique for each $\lambda > 0$, that is, there exists an inverse function for Q.

Lemma 3.2. *Q satisfies the following properties.*

(i) Q *is strictly decreasing.*

(ii) $Q(P) \to \infty$ *as* $P \to 0$.

(iii) $Q(P) = 0$ *if* P *satisfies* $(1+r)\delta u'((1+r)P) - u'(I-P)$.

Proof of property (i). Let P and P' be arbitrary premiums such that $P' > P$. By the assumptions of $u'(c) > 0$ and $u''(c) < 0$, we have

$$u'(I-P') - \frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P'\right) > u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P\right)$$

and

$$(1+r)\delta u'((1+r)P') - u'(I-P') < (1+r)\delta u'((1+r)P) - u'(I-P).$$

So it follows that

$$\begin{aligned} Q(P) = \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P\right)} &> \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P') - \frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P'\right)} \\ &> \frac{(1+r)\delta u'((1+r)P') - u'(I-P')}{u'(I-P') - \frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P'\right)} = Q(P'). \end{aligned}$$

Therefore, Q is strictly decreasing. □

Proof of property (ii). By assumption 2.3, we have

$$\lim_{P\to 0} Q(P) = \lim_{P\to 0}\left[\frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P\right)}\right] = \infty.$$

□

Proof of property (iii). Since $u'(I-P)-\frac{\rho-r}{1+\rho}u'\left(I-\frac{\rho-r}{1+\rho}P\right)$ is positive for any $P \leqslant I < \infty$, the property is obvious. □

The following theorem characterizes the optimal pension plan.

Theorem 3.1. *The optimal pension plan is determined by the inverse function of Q.*

Proof. Q corresponds to an arbitrary pension premium $P \in \mathscr{P}$ to type $\lambda \in \Lambda$ whose optimal pension premium is P. By the lemma 3.2, Q is one to one function. In addition, properties (i) and (ii) say that Q is onto function. Therefore, for Q, there exists an inverse function that determines the optimal pension premium for any $\lambda \in \Lambda$. □

Theorem 3.1 together with property (i) in the lemma 3.2 says that it is optimal to set a lower premium for consumers that feel greater temptation. This result can be understood by the following discussion. Consider the situation when a consumer does not save and not borrow at all. His welfare function is rewritten as

$$\begin{aligned} W(B(\tau);\lambda) =&(1+\lambda)u(I-P)+\delta u((1+r)P)-\lambda u\left(I-\frac{\rho-r}{1+\rho}P\right) \\ =&u(I-P)+\delta u((1+r)P)-\lambda\left[u\left(I-\frac{\rho-r}{1+\rho}P\right)-u(I-P)\right]. \end{aligned}$$

The summation of the first and second terms represents the utility of consumption and the third term represents the cost of self-control. Differentiate both sides with P and we have the marginal utility minus the marginal cost of P,

$$\begin{aligned} \frac{\partial W(B(\tau);\lambda)}{\partial P} =&-u'(I-P)+(1+r)\delta u'((1+r)P) \\ &-\lambda\left[-u'\left(I-\frac{\rho-r}{1+\rho}P\right)\left(\frac{\rho-r}{1+\rho}\right)+u'(I-P)\right]. \end{aligned}$$

The first-order condition says that the marginal utility equals the marginal cost at optimal P. Differentiate this again with P and we have

$$\begin{aligned} \frac{\partial^2 W(B(\tau);\lambda)}{\partial P^2} =&u''(I-P)+(1+r)^2\delta u''((1+r)P) \\ &-\lambda\left[u''\left(I-\frac{\rho-r}{1+\rho}P\right)\left(\frac{\rho-r}{1+\rho}\right)^2-u''(I-P)\right]. \end{aligned}$$

By assumption 2.2, the marginal utility decreases as P increases. And in the proof of the lemma 3.2, we have seen that the marginal cost increases as P does. Note that the increase in λ raises the marginal cost and we can see that P, at the point where marginal utility equals marginal cost, moves lower. Therefore, higher λ implies lower $P(\lambda)$. Intuitively, the increase in P has two effects. One effect reduces the amount of money the consumer can use when she is young. For the optimal plan, it holds that $R=(1+r)P$. Hence the increase of P means the increase of R. Thus another effect enlarges the amount of money that the consumer can borrow when she is young and it increases the temptation of borrowing. For consumers with larger λ, the latter effect is stronger. Therefore to avoid a high self-control cost, it is not proper to apply a large pension premium to a consumer with large λ. This result can be understood also by a picture. In Figure 2, each of three indifference curves for τ_L, τ_M, τ_H, where ,$\tau_L < \tau_M < \tau_H$, is a tangent to line $P=\frac{R}{1+r}$. The tangent points are the optimal plans. Obviously, as λ increases, the optimal plan is determined in the lower-left area of the plane.

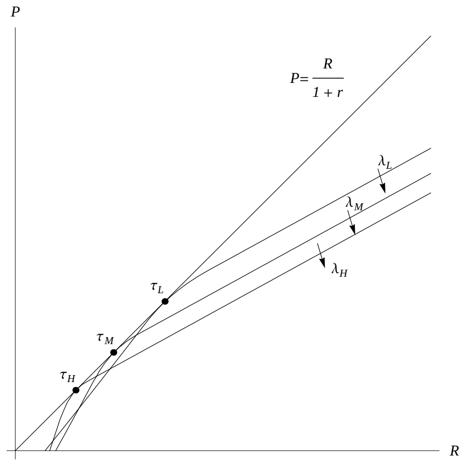

Figure 2: Optimal plans

At the end of this subsection, we consider the value of $(1+r)\delta u'((1+r)P) - u'(I-P)$ in property (iii). It is equivalent to the amount of saving when $\lambda = 0$ and there is no pension policy. We have already shown that any $\tau \in T_s^0$ is optimal for a consumer with $\lambda = 0$. That is, when a consumer does not feel any temptation, it is optimal to impose an arbitrary premium between 0 and the optimal amount of saving. This is a well-known result: if the pension earns the same interest rate as private savings, these two ways of saving are indifferent and so the pension does not improve welfare (For example, see **?**). Interestingly, however, when a consumer has $\lambda > 0$, this result does not hold. Even if the interest rates are equal to each other, the private saving and the pension are not equivalent.

3.2 Two types of consumers

In this section, we consider two types of heterogeneous consumers. The assumption of complete information is a useful benchmark. So we begin with the situation where the government knows what type each consumer has. Denote the types as λ_L and λ_H, where $0 < \lambda_L < \lambda_H$. And the population of λ_L and λ_H are n_L and n_H, respectively, where we assume that $n_L > 0$, $n_H > 0$ and $n_L + n_H = 1$. Note that, unlike in the identical-type case, we cannot state that $R = (1+r)P$ is always satisfied at optimal schedules since there may be transfer between the types.

Remember that the marginal welfare of P for each agent is negative. Hence the pension schedule $S \equiv \{\tau_L, \tau_H\}$ such that $P_L > \frac{R_L}{1+r}$ and $P_H > \frac{R_H}{1+r}$ is not optimal since the schedule satisfies the feasibility constraint slackly, so that a sufficiently small decrease in P is feasible and improves social welfare. It is also clear that the pension schedule such that $P_L < \frac{R_L}{1+r}$ and $P_H < \frac{R_H}{1+r}$ is not feasible.

In addition, the following lemma holds.

Lemma 3.3. *If a pension schedule $\{\hat{\tau}_L, \hat{\tau}_H\}$ is feasible, then the schedules $\{\tau'_L, \tau'_H\}$ such that*

$$\left[P'_L \geqslant \frac{R'_L - \hat{R}_L}{1+r} + \hat{P}_L, \tau'_H = \hat{\tau}_H \right] \tag{8}$$

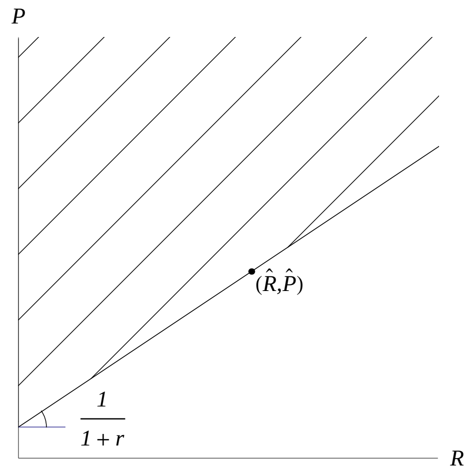

Figure 3: Individually feasible schedules for $(\hat{R}, \hat{P})$

or

$$\left[P'_H \geqslant \frac{R'_H - \hat{R}_H}{1+r} + \hat{P}_H, \tau'_L = \hat{\tau}_L\right] \tag{9}$$

are also feasible.

Proof. Without loss of generality, we show that the schedule that satisfies (8) is feasible. Suppose that $\{\hat{\tau}_L, \hat{\tau}_H\}$ is feasible; that is,

$$\begin{aligned} &(1+r)(n_L\hat{P}_L + n_H\hat{P}_H) \geqslant n_L\hat{R}_L + n_H\hat{R}_H \\ \Leftrightarrow &(1+r)n_L\hat{P}_L \geqslant -(1+r)n_H\hat{P}_H + n_L\hat{R}_L + n_H\hat{R}_H. \end{aligned} \tag{10}$$

Choose a schedule such that $P'_L \geqslant \frac{R'_L - \hat{R}_L}{1+r} + \hat{P}_L, \tau'_H = \hat{\tau}_H$ arbitrarily. Then it follows that

$$\begin{aligned} &P'_L \geqslant \frac{R'_L - \hat{R}_L}{1+r} + \hat{P}_L \\ \Leftrightarrow &(1+r)n_LP'_L - n_L(R'_L - \hat{R}_L) \geqslant (1+r)n_L\hat{P}_L. \end{aligned}$$

Together with (10), we have

$$\begin{aligned} &(1+r)n_LP'_L - n_L(R'_L - \hat{R}_L) + (1+r)n_H\hat{P}_H \geqslant n_L\hat{R}_L + n_H\hat{R}_H \\ \Leftrightarrow &(1+r)(n_LP'_L + n_H\hat{P}_H) \geqslant n_LR'_L + n_H\hat{R}_H. \end{aligned}$$

By $\tau'_H = \hat{\tau}_H$, it follows that

$$(1+r)(n_Lp'_L + n_HP'_H) \geqslant n_LR'_L + n_HR'_H.$$

Thus, the schedule $\{\tau'_L, \tau'_H\}$ is feasible. We can show the feasibility of the schedule that satisfies (9). □

For a given plan $(\hat{\tau}_\lambda)$, we call the plans τ_λ that satisfy $P_\lambda \geqslant \frac{R_\lambda - \hat{R}_\lambda}{1+r} + \hat{P}_\lambda$ individually feasible. Figure 3 depicts the set of schedules that is individually feasible for $(\hat{R}, \hat{P})$. The lemma 3.3 is useful for searching for solutions. For example, in Figure 4, $\sigma_\lambda(\tau)$ is lower

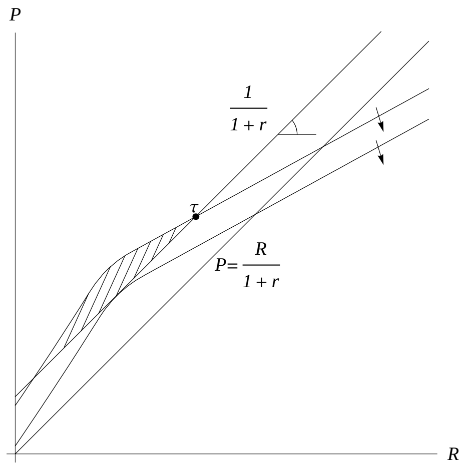

Figure 4: τ is not optimal plan

than $\frac{1}{1+r}$. Note that with complete information, we can ignore the incentive compatibility conditions. Hence a pension plan for another type can be fixed arbitrarily. Then there is room for the improvement. Indeed, obviously τ_λ is not optimal since we can find individually feasible and more preferable plans in the hatched area in the Figure. The point is that if this improvement is available, it works individually; that is, it does not need transfer between types. Thus this is a Pareto improvement.

The next lemma is important for proving the lemma 3.5.

Lemma 3.4. *For any $\lambda \in \Lambda \subseteq \mathbb{R}_+$, τ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ satisfies $R > 0$.*

Proof. Fix arbitrary $\lambda \in \Lambda \subseteq \mathbb{R}_+$ and choose τ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$. Note that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ implies $\tau \in T_b$. $\sigma_\lambda(\tau)$ is equal to

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}.$$

Then, by assumption 2.3,

$$\begin{aligned}
&\lim_{R\to 0} \frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \\
=&\frac{-\lim_{R\to 0} \delta u'(R) + \frac{\lambda}{1+\rho} u'(I-P)}{-u'(I-P)} \\
=&\frac{-\infty}{-u'(I-P)} = \infty.
\end{aligned}$$

Since $\sigma_\lambda(\tau)$ is decreasing in R, τ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ satisfies $R > 0$. □

The following lemma, derived from the previous lemma gives the condition for optimality.

Lemma 3.5. *In the optimal schedule with complete information $\{\tau_L, \tau_H\}$, both $\sigma_L(\tau_L)$ and $\sigma_H(\tau_H)$ are equal to $\frac{1}{1+r}$.*

Proof. Suppose that either $\sigma_L(\tau_L)$ or $\sigma_H(\tau_H)$ is not equal to $\frac{1}{1+r}$. Without loss of generality, suppose that $\sigma_L(\lambda_L) \neq \frac{1}{1+r}$.

(i) $\sigma_L(\tau_L) > \frac{1}{1+r}$

Consider ε_1, $\varepsilon_2 > 0$ such that $\varepsilon_2 = \frac{\varepsilon_1}{1+r}$. For such ε_1 and ε_2, let $R' \equiv R_L + \varepsilon_1$ and $P' \equiv P_L + \varepsilon_2$. Then it follows that

$$\begin{aligned} \varepsilon_2 &= \frac{\varepsilon_1}{1+r} \\ \Leftrightarrow P_L + \varepsilon_2 &= \frac{R_L + \varepsilon_1 - R_L}{1+r} + P_L \\ \Leftrightarrow P' &= \frac{R' - R_L}{1+r} + P_L, \end{aligned}$$

that is, $\tau' = (R', P')$ is individually feasible by lemma 3.3. The line represented by this equation has the slope of $\frac{1}{1+r}$ on R-P plane and passes through (R_L, P_L). Furthermore, the marginal rate of substitution at τ_L is $\sigma_L(\tau_L) > \frac{1}{1+r}$. Thus for sufficiently small ε_1 and ε_2, τ' brings more individual welfare. Since there exists an another schedule that is feasible and preferable to $\{\tau_L, \tau_H\}$, $\{\tau_L, \tau_H\}$ is not optimal.

(ii) $\sigma_L(\tau_L) < \frac{1}{1+r}$

Consider ε_1, $\varepsilon_2 > 0$, which satisfies that $\varepsilon_2 = \frac{\varepsilon_1}{1+r}$. For such ε_1 and ε_2, let $R' \equiv R_L - \varepsilon_1$ and $P' \equiv P_L - \varepsilon_2$. Here, for τ_L such that $\sigma_L(\tau_L) < \frac{1}{1+r}$, it is satisfied that $R > 0$ by the lemma 3.4. Then, similar to the previous case, it is satisfied that

$$\begin{aligned} \varepsilon_2 &= \frac{\varepsilon_1}{1+r} \\ \Leftrightarrow P_L - \varepsilon_2 &= \frac{R_L - \varepsilon_1 - R_L}{1+r} + P_L \\ \Leftrightarrow P' &= \frac{R' - R_L}{1+r} + P_L, \end{aligned}$$

that is, $\tau' = (R', P')$ is individually feasible by the lemma 3.3. This line has the slope of $\frac{1}{1+r}$ and passes through (R_L, P_L). Furthermore, the marginal rate of substitution at τ_L is $\sigma_L(\tau_L) < \frac{1}{1+r}$. Thus, for sufficiently small ε_1 and ε_2, τ' brings more individual welfare. Since there exists another schedule that is feasible and preferable to $\{\tau_L, \tau_H\}$, $\{\tau_L, \tau_H\}$ is not optimal. □

By the lemma 3.5, if a pension schedule is optimal, it holds that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ for all $\lambda \in \Lambda$. Together with the lemma 3.4, if a pension schedule is optimal, we have $R_\lambda > 0$ for all $\lambda \in \Lambda$. That is, if a consumer feels temptation, the funded pension improves social welfare.

Here, we have the following result.

Theorem 3.2. *If $\lambda \neq \lambda'$, it is not optimal that $\tau_\lambda = \tau_{\lambda'}$.*

Proof. Consider an arbitrary λ, λ'. By the lemma 3.5, if the singleton schedule of $\{\tau\}$ is optimal, it is satisfied that

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{1}{1+r}$$

and

$$\frac{-\delta u'(R) + \frac{\lambda'}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda')u'(I-P) + \lambda' u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{1}{1+r},$$

then it must be follow that

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{-\delta u'(R) + \frac{\lambda'}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda')u'(I-P) + \lambda' u'\left(I - P + \frac{R}{1+\rho}\right)}. \tag{11}$$

This is a necessary condition for τ is optimal. For simplicity, we abbreviate some parts of (11) as follows:

$$\begin{aligned}
&A \equiv -\delta u'(R)\\
&B \equiv \frac{1}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)\\
&C \equiv u'(I-P)\\
&D \equiv u'(I-P) + u'\left(I - P + \frac{R}{1+\rho}\right).
\end{aligned}$$

Then (11) is rewritten to,

$$\begin{aligned}
&\frac{A+\lambda B}{C+\lambda D} = \frac{A+\lambda' B}{C+\lambda' D}\\
\Leftrightarrow &(A+\lambda B)(C+\lambda' D) = (A+\lambda' B)(C+\lambda D)\\
\Leftrightarrow &(\lambda - \lambda')(AD - BC) = 0.
\end{aligned}$$

By assumption 2.2, $AD = -\delta u'(R)\left[u'(I-P) + u'\left(I - P + \frac{R}{1+\rho}\right)\right]$ is strictly negative and $BC = \frac{1}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right) u'(I-P)$ is strictly positive, it must be that $\lambda = \lambda'$. This implies (11), necessary condition for singleton schedule to be optimal, is satisfied only if $\lambda = \lambda'$. □

That is, if consumers have different strength of temptation, applying a common pension plan is not optimal. This is consistent with our first intuition.

Next we construct the optimal schedule in this situation. To obtain a precise result, henceforth we specialize in the normative utility function as $u(c) = \log c$ for all $c \in \mathbb{R}_+$. $\log c$ satisfies assumptions 2.2 and 2.3. For this specialized utility function, we have the following consumptions:

$$c_1(\tau) = \begin{cases} \frac{(1+\lambda)\left((1+r)(I-P)+R\right)}{(1+r)(1+\delta+\lambda)} & \text{if } 0 \leqslant P < I - \frac{(1+\lambda)R}{(1+r)\delta}\\ I - P & \text{if } I - \frac{(1+\lambda)R}{(1+r)\delta} \leqslant P \leqslant I - \frac{(1+\lambda)R}{(1+\rho)\delta}\\ \frac{(1+\lambda)\left((1+\rho)(I-P)+R\right)}{(1+\rho)(1+\delta+\lambda)} & \text{otherwise,} \end{cases}$$

$$c_2(\tau) = \begin{cases} \frac{\delta\left((1+r)(I-P)+R\right)}{1+\delta+\lambda} & \text{if } 0 \leqslant P < I - \frac{(1+\lambda)R}{(1+r)\delta}\\ R & \text{if } I - \frac{(1+\lambda)R}{(1+r)\delta} \leqslant P \leqslant I - \frac{(1+\lambda)R}{(1+\rho)\delta}\\ \frac{\delta\left((1+\rho)(I-P)+R\right)}{1+\delta+\lambda} & \text{otherwise.} \end{cases}$$

Next we consider the locus of points at which $\sigma_\lambda(\tau)$ equals $\frac{1}{1+r}$. According to the lemma 3.1, it is enough to obtain the locus to consider τ such that $I - \frac{(1+\lambda)R}{(1+r)\delta} \leqslant P \leqslant I - \frac{(1+\lambda)R}{(1+\rho)\delta}$. Consider an arbitrary $\lambda \in \Lambda$. Then the individual welfare for such τ is

$$W(B(\tau);\lambda) = (1+\lambda)\log(I-P) + \delta\log(R) - \lambda\log\left(I - P + \frac{R}{1+\rho}\right).$$

We have

$$\sigma_\lambda(\tau) = \frac{-\frac{\delta}{R} + \frac{\lambda}{(1+\rho)(I-P)+R}}{-\frac{1+\lambda}{I-P} + \frac{(1+\rho)\lambda}{(1+\rho)(I-P)+R}}.$$

The relation between P and R such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ is as follows:

$$\begin{aligned} P =& I - K_\lambda R, \\ K_\lambda \equiv& \frac{1+\rho+(1+r)(\lambda-\delta)+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda)^2+2\delta(1+r)(1+(1-r)\lambda+(1+2\lambda)\rho)}}{2\delta(1+r)(1+\rho)}. \end{aligned}$$

Proposition 3.1. *For any $\lambda > 0$ and $\lambda' > 0$ such that $\lambda \neq \lambda'$, if $\sigma_\lambda(\tau) = \sigma_{\lambda'}(\tau') = \frac{1}{1+r}$, then $\tau \neq \tau'$.*

Proof. Fix arbitrarily $\lambda > 0$ and $\lambda' > 0$ such that $\lambda \neq \lambda'$. Suppose that for τ and τ', $\sigma_\lambda(\tau) = \sigma_{\lambda'}(\tau') = \frac{1}{1+r}$. Then it holds that $P = I - K_\lambda R$ and $P' = I - K_{\lambda'} R'$. Note that the relation between P and R and that between P' and R' are linear with the slopes of $-K_\lambda$ and $-K_{\lambda'}$, respectively. We show that if $\lambda > \lambda'$, then $K_\lambda > K_{\lambda'}$. Calculating $K_\lambda - K_{\lambda'}$, we have

$$\begin{aligned} & K_\lambda - K_{\lambda'} \\ =& \frac{1+\rho+(1+r)(\lambda-\delta)+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda)^2+2\delta(1+r)(1+(1-r)\lambda+(1+2\lambda)\rho)}}{2\delta(1+r)(1+\rho)} \\ & - \frac{1+\rho+(1+r)(\lambda'-\delta)+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda')^2+2\delta(1+r)(1+(1-r)\lambda'+(1+2\lambda')\rho)}}{2\delta(1+r)(1+\rho)} \\ =& \frac{(1+r)\lambda+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda)^2+2\delta(1+r)(1+(1-r)\lambda+(1+2\lambda)\rho)}}{2\delta(1+r)(1+\rho)} \\ & - \frac{(1+r)\lambda'+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda')^2+2\delta(1+r)(1+(1-r)\lambda'+(1+2\lambda')\rho)}}{2\delta(1+r)(1+\rho)}. \end{aligned}$$

Comparing inside the square roots, the former is larger than the latter. Thus we have $K_\lambda > K_{\lambda'}$. Though $\tau = \tau'$ holds only if $R = R' = 0$, $R > 0$ and $R' > 0$ by the lemma 3.4. Therefore $\tau \neq \tau'$. □

This proposition states that the point at which the indifference curve of each consumer is tangential to the feasibility frontier differs according to the type.

By the lemma 3.5, the optimal schedule satisfies that $\sigma_L(\tau_L) = \sigma_H(\tau_H) = \frac{1}{1+r}$ so we have

$$\begin{aligned} P_L &= I - K_L R_L \\ P_H &= I - K_H R_H. \end{aligned}$$

As we have mentioned, the feasibility condition is satisfied with equality for the optimal schedule. Hence, according to the feasibility condition, we have

$$
\begin{aligned}
&(1+r)[n_L P_L + n_H P_H] = n_L R_L + n_H R_H \\
\Leftrightarrow &(1+r)[n_L(I - K_L R_L) + n_H(I - K_H R_H)] = n_L R_L + n_H R_H \\
\Leftrightarrow &(1+r)[I - n_L K_L R_L - n_H K_K R_H] = n_L R_L + n_H R_H \\
\Leftrightarrow &(1+(1+r)K_H) n_H R_H = (1+r)I - (1+(1+r)K_L) n_L R_L \\
\Leftrightarrow &R_H = \frac{(1+r)I}{(1+(1+r)K_H)n_H} - \frac{(1+(1+r)K_L)n_L}{(1+(1+r)K_H)n_H} R_L.
\end{aligned}
$$

Thus the summarized problem is that

$$
\begin{aligned}
\max_{\{\tau_\lambda\}_{\lambda=L,H}} \; & n_L \left[(1+\lambda_L)\log(I - P_L) + \delta \log R_L - \lambda_L \log\left(I - P_L + \frac{R_L}{1+r\rho}\right)\right] \\
& + n_H \left[(1+\lambda_H)\log(I - P_H) + \delta \log R_H - \lambda_H \log\left(I - P_H + \frac{R_H}{1+r\rho}\right)\right] \\
\text{s.t. } & P_L = I - K_L R_L, \; P_H = I - K_H R_H \\
& R_H = \frac{(1+r)I}{(1+(1+r)K_H)n_H} - \frac{(1+(1+r)K_L)n_L}{(1+(1+r)K_H)n_H} R_L.
\end{aligned}
$$

Substituting P_L, P_H, and R_H, this problem can be seen as simply one variable problem. The first-order condition is

$$
\begin{aligned}
\frac{\partial W}{\partial R_L} = & \frac{n_L[(n_H + n_L)(1+(1+r)K_L)R_L - (1+r)I](1+\delta)}{R_L(n_L(1+(1+r)K_L)R_L - (1+r)I)} = 0 \\
\Leftrightarrow & R_L = \frac{(1+r)I}{1+(1+r)K_L}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
P_L = & I - \frac{K_L(1+r)I}{1+(1+r)K_L} = \frac{I}{1+(1+r)K_L} \\
R_H = & \frac{(1+r)I - (1+(1+r)K_L)n_L \frac{(1+r)I}{(1+(1+r)K_L)}}{(1+(1+r)K_H)n_H} = \frac{(1+r)I}{1+(1+r)K_H}. \\
P_H = & I - K_H \frac{(1+r)I}{1+(1+r)K_H} = \frac{I}{1+(1+r)K_H}.
\end{aligned}
$$

Seeing this result, we find that the optimal schedule satisfies $R_\lambda = (1+r)P_\lambda$ for $\lambda \in \Lambda$. In this situation, the incentive compatibility condition is strictly satisfied. Eventually, this way of determining optimal plans is the same as in the case of the unique type. Thus, function $Q : [0,1] \to \Lambda$, defined in a previous chapter, characterizes the optimal schedule. With specialization of the normative utility function, we have

$$
\begin{aligned}
Q(x) = & \frac{(1+r)\delta u'((1+r)x) - u'(I-x)}{u'(I-x) - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho-r}{1+\rho}x\right)} \\
= & \frac{\frac{\delta}{x} - \frac{1}{I-x}}{\frac{1}{I-x} - \frac{\rho-r}{1+\rho}\frac{1}{I - \frac{\rho-r}{1+\rho}x}}.
\end{aligned}
$$

Letting $Q(x) = \lambda$ and solving this equation with x, we have

$$\begin{aligned} x =& \frac{(1+\delta)(1+\rho)+\delta(\rho-r)+(1+r)\lambda}{2(1+\delta)(\rho-r)} I \\ &- \frac{\sqrt{\left((1+\delta)(1+\rho)+\delta(\rho-r)+(1+r)\lambda\right)^2 - 4\delta(1+\delta)(1+\rho)(\rho-r)}}{2(1+\delta)(\rho-r)} I. \end{aligned}$$

Therefore, we have the following theorem.

Theorem 3.3. *Assume $u(c) = \log c$ and $|\Lambda| = 2$. The optimal pension schedule has the following form: for any $\lambda \in \Lambda$,*

$$\begin{aligned} P_\lambda =& \frac{(1+\delta)(1+\rho)+\delta(\rho-r)+(1+r)\lambda}{2(1+\delta)(\rho-r)} I \\ &- \frac{\sqrt{\left((1+\delta)(1+\rho)+\delta(\rho-r)+(1+r)\lambda\right)^2 - 4\delta(1+\delta)(1+\rho)(\rho-r)}}{2(1+\delta)(\rho-r)} I, \\ R_\lambda =& (1+r)P_\lambda. \end{aligned}$$

We can see some things from the result. First, there is no monetary transfer among types since the optimal plans are balanced for each type. Second, similar to the identical-type case, there are no private saving and borrowing at the optimal schedule. This implies that the monetary market is balanced. Third, and importantly, the optimal schedule does not depend on the distribution of types.Indeed, it does not contain n_L and n_H. Furthermore, consider the following mechanism. The government shows a pension schedule such that

$$\left\{(R, P) \in T \colon P = \frac{R}{1+r}\right\}$$

and let each consumer announce the maximal amount of pension she foresees wanting in old age. Then the consumer is enrolled in the pension plan that is in accord with her decision. Though formal proof is omitted, this mechanism implements the optimal schedule with a weakly dominant strategy. This follows by the convexity of individual welfare and not by the existing externality between types. Note that this mechanism does not need any information about types, such as the distribution and even what types there are.

4 Generalization

In this section, we generalize our model. One generalization is about the number of types. First, we consider the case of many finite types. Then we consider the case of continuous types. As in the previous chapter, we specialize the normative utility function as a logarithm function.

4.1 Finite types

Consider the set of types $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, where $m < \infty$. Denote the population of type λ_i by n_i, where it is assumed that $\sum_{i=1}^m n_i = 1$ and $n_i > 0$ for all $\lambda_i \in \Lambda$. The problem is that

$$\begin{aligned}
&\max_S \sum_{i=1}^m n_i \left\{(1+\lambda_i)\log(I-P_i) + \delta \log R_i - \lambda_i \log\left(I - P_i + \frac{R_i}{1+\rho}\right)\right\}\\
&\text{s.t. } (1+r)\sum_{i=1}^m n_i P_i \geqslant \sum_{i=1}^m n_i R_i.
\end{aligned}$$

We can apply the lemma 3.5 also in this model. Thus, by substituting $P_i = I - K_i R_i$ in the problem, we have

$$\begin{aligned}
&\max_S \sum_{i=1}^m n_i \left\{(1+\lambda_i)\log K_i R_i + \delta \log R_i - \lambda_i \log\left(\left(K_i + \frac{1}{1+\rho}\right) R_i\right)\right\}\\
&\text{s.t. } (1+r)\sum_{i=1}^m n_i (I - K_i R_i) \geqslant \sum_{i=1}^m n_i R_i.
\end{aligned}$$

The associated Lagrangian is

$$\begin{aligned}
\mathscr{L} \equiv &\mu_0 \left[\sum_{i=1}^m n_i \left\{(1+\lambda_i)\log K_i R_i + \delta \log R_i - \lambda_i \log\left(\left(K_i + \frac{1}{1+\rho}\right) R_i\right)\right\}\right]\\
&+ \mu_1 \left[(1+r)\sum_{i=1}^m n_i (I - K_i R_i) - \sum_{i=1}^m n_i R_i\right].
\end{aligned}$$

The necessary condition is

$$\begin{aligned}
&\frac{\partial \mathscr{L}}{\partial R_j} = \mu_0 \left[n_j \left\{\frac{1+\lambda_i}{R_j} + \frac{\delta}{R_j} - \frac{\lambda_j}{R_j}\right\}\right] + \mu_1[-(1+r)n_j K_j - n_j] = 0, \ \forall \lambda \in \Lambda\\
&(\mu_0, \mu_1) \geqslant 0\\
&(\mu_0, \mu_1) \neq 0\\
&\mu_1 \left[(1+r)\sum_{i=1}^m n_i (I - K_i R_i) - \sum_{i=1}^m n_i R_i\right] = 0.
\end{aligned}$$

Note that $[-(1+r)n_j K_j - n_j] < 0$. Then if $\mu_0 = 0$, the first order condition is satisfied only if $\mu_1 = 0$. This contradicts the non-zero condition of Lagrange multipliers. Thus $\mu_0 > 1$ and we can standardize this as $\mu_0 = 1$. Then the first-order conditions are rearranged as

$$n_j \frac{1+\delta}{R_j} - \mu_1[(1+r)n_j K_j - n_j] = 0, \ \forall \lambda \in \Lambda. \tag{12}$$

μ_1 is not zero since the first term is strictly positive. So the feasibility condition is satisfied with equality. By (12), we have

$$\mu_1 = \frac{1}{(1+r)K_j + 1}\left(\frac{1+\delta}{R_j}\right).$$

Thus for any $\lambda_j \in \Lambda$ and $\lambda_k \in \Lambda$, it follows that

$$\begin{aligned}
&\frac{1}{(1+r)K_j + 1}\left(\frac{1+\delta}{R_j}\right) = \frac{1}{(1+r)K_k + 1}\left(\frac{1+\delta}{R_k}\right)\\
\Leftrightarrow &[(1+r)K_j + 1]R_j = [(1+r)K_k + 1]R_k. \qquad (13)
\end{aligned}$$

By the feasibility condition with equality, we have

$$\begin{aligned}(1+r)\sum_{i=1}^{m} n_i(I - K_iR_i) - \sum_{i=1}^{m} n_iR_i = 0 \\ \Leftrightarrow \sum_{i=1}^{m} n_i((1+r)K_i + 1)R_i = (1+r)I\sum_{i=1}^{m} n_i = (1+r)I.\end{aligned}$$

Fixing arbitrarily $\lambda_j \in \Lambda$ and using (13), it follows that

$$\begin{aligned}(1+r)I =& \sum_{i=1}^{m} n_i((1+r)K_i + 1)R_i = ((1+r)K_j + 1)R_j \sum_{i=1}^{m} n_i \\ =& ((1+r)K_j + 1)R_j \\ \Leftrightarrow R_j =& \frac{(1+r)I}{(1+r)K_j + 1}.\end{aligned}$$

Substituting this to $P_j = I - K_jR_j$, we have

$$P_j = I - K_j\frac{(1+r)I}{(1+r)K_j + 1} = \frac{((1+r)K_j + 1)I - (1+r)K_jI}{(1+r)K_j + I} = \frac{I}{(1+r)K_j + I}.$$

Thus in the solution $R_j = (1+r)P_j$ is satisfied for any type $\lambda_j \in \Lambda$. This is the same result as in the two types case.

4.2 Continuous types

Let $\Lambda = [\underline{\lambda}, \overline{\lambda}]$, where $\underline{\lambda} \geqslant 0$. Each λ is independently and identically distributed on Λ according to the distribution function F. $(R(\lambda), P(\lambda))$ denotes the pension plan for the consumer whose type is $\lambda \in \Lambda$.

Similarly, considering the problem with complete information, it follows that $P(\lambda) = I - K(\lambda)R(\lambda)$, $c_1^{\lambda} = I - P(\lambda)$ and $c_2^{\lambda} = R(\lambda)$ for each λ, where $K(\lambda)$ is the constant that has the same form as K_λ in the previous chapter. The problem is that

$$\begin{aligned}&\max_{S} \int_{\underline{\lambda}}^{\overline{\lambda}} \left[(1+\lambda)\log(I - P(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right)\right] dF(\lambda) \\ &\text{s.t.} \int_{\underline{\lambda}}^{\overline{\lambda}} [(1+r)P(\lambda) - R(\lambda)]\, dF(\lambda) = 0 \\ &\quad P(\lambda) = I - K(\lambda)R(\lambda),\end{aligned}$$

where we can assume that the solution satisfies the feasibility condition with equality for the same reason as in the previous chapter. Since the feasibility constraint includes a integration, we rewrite it. We define a new state variable h as

$$h(\lambda) \equiv \int_{\underline{\lambda}}^{\lambda} [(1+r)P(x) - R(x)]\, dF(x).$$

Then, h satisfies the following conditions, conversely implying the feasibility condition.

$$\begin{aligned}h'(\lambda) =& (1+r)P(\lambda) - R(\lambda), \\ h(\underline{\lambda}) =& 0 \text{ and } h(\overline{\lambda}) = 0.\end{aligned}$$

Thus, we have the rewritten problem:

$$\max_S \int_{\underline{\lambda}}^{\overline{\lambda}} \left[(1+\lambda)\log(I-P(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right)\right] dF(\lambda)$$
$$\text{s.t. } h'(\lambda) = (1+r)P(\lambda) - R(\lambda),$$
$$h(\underline{\lambda}) = 0 \text{ and } h(\overline{\lambda}) = 0$$
$$P(\lambda) = I - K(\lambda)R(\lambda).$$

The associated Hamiltonian is,

$$\begin{aligned}\mathscr{H} \equiv &(1+\lambda)\log(I-P(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right) + \mu(\lambda)h'(\lambda)\\ =&(1+\lambda)\log(I-P(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right)\\ &+ \mu(\lambda)\left[(1+r)P(\lambda) - R(\lambda)\right],\end{aligned}$$

where μ is the co-state variable. Substituting $P(\lambda) = I - K(\lambda)R(\lambda)$, we have,

$$\begin{aligned}\mathscr{H} =&(1+\lambda)\log(K(\lambda)R(\lambda)) + \delta\log(R(\lambda)) - \lambda\log\left(\left(K(\lambda) + \frac{1}{1+\rho}\right)R(\lambda)\right)\\ &+ \mu(\lambda)\left[(1+r)I - ((1+r)K(\lambda)+1)R(\lambda)\right].\end{aligned}$$

By Pontryagin's principle,

$$\begin{aligned}\frac{\partial\mathscr{H}}{\partial R(\lambda)} =&\frac{1+\lambda}{R(\lambda)} + \frac{\delta}{R(\lambda)} - \frac{\lambda}{R(\lambda)} - \mu(\lambda)((1+r)K(\lambda)+1)R(\lambda)\\ =&\frac{1+\delta}{R(\lambda)} - \mu(\lambda)((1+r)K(\lambda)+1 = 0 \qquad (14)\\ \mu'(\lambda) =& -\frac{\partial\mathscr{H}}{\partial h(\lambda)}.\end{aligned}$$

Note that $-\frac{\partial\mathscr{H}}{\partial h(\lambda)} = 0$, so $u'(\lambda) = 0$. This implies that $u(\lambda)$ does not depend on λ. Hence we can simply write $\mu(\lambda) = \mu$. (14) can be rearranged as,

$$R(\lambda) = \frac{1+\delta}{((1+r)K(\lambda)+1)\mu}.$$

By the feasibility condition,

$$\begin{aligned}&\int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)(I - K(\lambda)R(\lambda)) - R(\lambda)dF(\lambda)\\ =&\int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)I - ((1+r)K(\lambda)+1)\, R(\lambda)dF(\lambda)\\ =&\int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)I - ((1+r)K(\lambda)+1)\, \frac{1+\delta}{((1+r)K(\lambda)+1)\,\mu} dF(\lambda)\\ =&\left((1+r)I - \frac{1+\delta}{\mu}\right)\int_{\underline{\lambda}}^{\overline{\lambda}} dF(\lambda) = (1+r)I - \frac{1+\delta}{\mu} = 0\\ \Leftrightarrow \mu =&\frac{1+\delta}{(1+r)I}\end{aligned}$$

Therefore we have,

$$
\begin{aligned}
R(\lambda) =& \frac{1+\delta}{((1+r)K(\lambda)+1)\mu} = \frac{1+\delta}{((1+r)K(\lambda)+1)\frac{1+\delta}{(1+r)I}} = \frac{(1+r)I}{(1+r)K(\lambda)+1}, \\
P(\lambda) =& I - K(\lambda)R(\lambda) = I - \frac{(1+r)K(\lambda)I}{(1+r)K(\lambda)+1} = \frac{I}{(1+r)K(\lambda)+1}.
\end{aligned}
$$

Again we obtained the relation $R(\lambda) = (1+r)P(\lambda)$. For the same reason as in the previous chapter, if we construct the pension schedule of such plans, the IC condition is strictly satisfied. The following theorem summarizes this section.

Theorem 4.1. *Assume $u(c) = \log c$ and $\Lambda \subseteq \mathbb{R}_+$. The optimal pension schedule has the following form: for any $\lambda \in \Lambda$,*

$$
\begin{aligned}
P(\lambda) =& I - K(\lambda)R(\lambda) \\
R(\lambda) =& (1+r)P(\lambda).
\end{aligned}
$$

5 Discussion

5.1 The effect of a borrowing constraint

So far, we have assumed that consumers are allowed to borrow money in the first period. In this section, we consider the special case in which consumers can borrow no money; that is, we restrict $s \geqslant 0$[2]. In the context of self-control preference, this assumption has an important meaning. The impossibility of borrowing after paying a premium strengthens the funded pension scheme as a commitment device.

With the borrowing constraint in place, the budget constraint is simply

$$
B(\tau) = \{(c_1, c_2) \in \mathbb{R}_+^2 : c_1 + s \leqslant I - P, c_2 \leqslant (1+r)(I - c_1) + R\}.
$$

For simplicity, we specialize a normative utility function as $u(c) = \log(c)$. Then the consumption in period 1 is

$$
c_1(\tau) = \begin{cases} \frac{(1+\lambda)[(1+r)(I-P)+R]}{(1+r)(1+\delta+\lambda)} & \text{if } 0 \leqslant P < I - \frac{(1+\lambda)R}{\delta(1+r)} \\ I - P & \text{if } I - \frac{(1+\lambda)R}{\delta(1+r)} \leqslant P \leqslant I. \end{cases}
$$

Similar to the case in section 3.1, we assume the identical type. Then $R = (1+r)P$ follows.

Calculating an optimal pension plan, we have

$$
\begin{aligned}
P_\lambda &= \begin{cases} \text{any number } P \in [0, \frac{\delta I}{1+\delta}] & \text{if } \lambda = 0 \\ \frac{\delta I}{1+\delta} & \text{if } \lambda > 0 \end{cases} \\
R_\lambda &= (1+r)P_\lambda, \ \forall \lambda \geqslant 0.
\end{aligned}
$$

Note that P does not depend on the type if $\lambda > 0$. This optimal P is equal to optimal saving when there is no temptation and no pension policy. Intuitively, by the consumption decision above, a consumer who paid $\frac{\delta I}{1+\delta}$ consumes all of remaining money in period 1. Since $\frac{\delta I}{1+\delta}$ is the optimal saving, the optimal consumption includes all of the remaining money. Thus, naturally, welfare is maximized without the harm of temptation. The

[2] There may be various strengths of the constraint, but here we consider only the strongest borrowing constraint.

interest rate for the pension is the same as that for private saving, so there is no difference in the amount of payout between the pension and saving. However, the pension, which make her pay a premium in advance has a role as a commitment device. Saving does not have the role since a consumer decide how much to save after she faces the temptation. Furthermore, importantly, now the consumer is not allowed to borrow, so the effect of an increase in the premium further strengthens the budget set: there is no temptation to borrow. This result is very different from that in chapters 3 and 4.

5.2 Robustness against a universal domain

With the specialization of $u(c) = \log c$, we have obtained a mechanism that implements the optimal schedule for any type of consumer. We next ask whether this result holds for any other normative utility functions. It is difficult to calculate the solution explicitly for most of the utility functions that are often used, such as constant power, CRRA, and CARA. We are not aware of any report on the application of the theory of temptation that gives an explicit solution [3]. Therefore, in this section we consider the expansion of type space instead of specialization on the normative utility function.

Let $\mathscr{U}$ be the set of arbitrary utility functions that satisfy assumptions 2.1, 2.2 and 2.3. Define the type space Θ to be $\Lambda \times \mathscr{U}$, where $\Lambda \equiv \mathbb{R}_+$. Note that this expansion does not affect the result of the lemma 3.1. Consider the mechanism that implements the optimal schedule for any vector of types $\theta \in \Theta^n$.

We call the mechanism that makes a consumer choose only the plan she wants a *simple mechanism.* The following proposition holds almost trivially.

Proposition 5.1. *A simple mechanism implements the optimal schedule for universal domain Θ^n if and only if it has the following form:*

$$\left\{(R, P) \in T \colon P = \frac{R}{1+r}\right\}.$$

Proof. Suppose that a simple mechanism has the form in the statement of the proposition. Since the lemma 3.1 holds for any $\theta_i \in \Theta$, it is best to choose a plan such that $P_i = \frac{R_i}{1+r}$ for every consumer. Similar to what we showed in the previous chapter, this mechanism satisfies all constraints and maximizes social welfare.

Next, suppose that a mechanism includes the plan $\hat{\tau}$ in the region of $R > \frac{P}{1+r}$. Without a loss of generality, choose a consumer and name her 1. Let $\hat{\theta}_1 \in \Theta$ be the type with which the consumer chooses $\hat{\tau}$, and let $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_1) \in \Theta^n$. Then for $\hat{\theta}$, all consumers choose $\hat{\tau}$ and it follows that $\sum_{i\in N} R_i > \sum_{i\in} \frac{P_i}{1+r}$, violating feasibility. Suppose that a simple mechanism includes the plan $\bar{\tau}$ in the region of $R < \frac{P}{1+r}$. Let $\bar{\theta}_1 \in \Theta$ be the type with which the consumer chooses $\bar{\tau}$, and let $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_1) \in \Theta^n$. Then for $\bar{\theta}$, all consumers choose $\bar{\tau}$ and it follows that $\sum_{i\in N} R_i < \sum_{i\in N} \frac{P_i}{1+r}$. Note that if a schedule is not pareto optimal, it does not maximize social welfare. Thus, this mechanism cannot implement an optimal schedule. □

6 Conclusion

We have considered optimally funded pensions for consumers who face the temptation to overconsume and as well as for those who have enough self-control to withstand their

[3] Krusell et al. (2010) show the property of optimal taxation for the CRRA normative utility function without an explicit solution.

temptation. Since funded pensions tighten consumers' budgets , they can serve as commitment devices to avoid overconsumption. We applied the pension to an economy in which consumers have heterogeneous self-control. We showed that funded pensions can improve social welfare even if the interest rates they draw are the same as those for private saving. In addition, consumers do not save individually when they choose the pension plan that is optimal for them. Furthermore, interestingly, lower pension premium and lower pension payout are applied for a higher temptation economy. This result is related to borrowing constraints. In an identical-type economy, an increase in the premium leads to increased pension income, and this augments possibility of debt. Since consumers are tempted to overconsume, this works stronger for consumers who have strong temptations. This effect is greater than the benefit of strengthening the budget set. As a main result, we have considered an optimal pension schedule when there exist two or more types in the economy. In that situation, we show the necessary conditions for the optimal schedule. If the normative utility function is a logarithm, it is characterized in the same way as that for an economy with identical-type. We show that monetary transfer among types will not occur for the optimal schedule. An important result is that the optimal schedule does not depend on the distribution of types; that is, what the government has to know is only what types are in the economy. This make the operation of the pension policy easier.

In the future research, we face many problems. In the latter half of this paper, we focused on the normative utility function as a logarithm. The first thing to do is to show that the result holds in a more general utility function. However, this challenge contains an analytical problem. We have not found a way to solve the problem explicitly for utility functions that are frequently used in economics, such as constant power,CRRA and CARA. Some noveel methods are needed to work out this point.

In addition, we restrict the policy to funded pensions. But pay-as-you-go pension can be considered. By association, we can consider a model in which a consumer lives longer than the two periods used in this study. Then it may be natural to assume a consumer earns wages in each period. Moreover, if we assume multiple periods, consumers may have the opportunity to change their pension plans in each period. It will be interesting to see whether this change has a positive effect on social welfare. These problems may be complex and challenging.

Acknowledgments

I would like to thank Eiichi Miyagawa for his invaluable comments and encouragements. In addition, I am grateful to Yasuyuki Miyahara, Shuhei Morimoto, Noritsugu Nakanishi, Yoshikatsu Tatamitani, Hirofumi Yamamura and participants of seminars at Kobe university and Japan Economic Association meeting at Doshisha university for their many constructive comments. Kazuhiko Hashimoto interested me in the theory of self-control and often gave advice about the research.

References

Peter A Diamond. A framework for social security analysis. *Journal of Public Economics*, 8(3):275–298, 1977.

Faruk Gul and Wolfgang Pesendorfer. Temptation and self-control. *Econometrica*, 69(6): 1403–1435, 2001.

Faruk Gul and Wolfgang Pesendorfer. Self-control, revealed preference and consumption choice. *Review of Economic Dynamics*, 7(2):243 – 264, 2004.

Kevin XD Huang, Zheng Liu, and Qi Zhu. Temptation and self-control: Some evidence and applications. *Working Paper*, 367, 2007.

Per Krusell, Burhanettin Kuruşçu, and Anthony A. Smith. Temptation and taxation. *Econometrica*, 78(6):2063–2084, 2010.

Çağri S. Kumru and Athanasios C. Thanopoulos. Social security and self control preferences. *Journal of Economic Dynamics and Control*, 32(3):757 – 778, 2008.

David Laibson. Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics*, pages 443–477, 1997.

Kerstin Roeder. Optimal taxes and pensions with myopic agents. *Social Choice and Welfare*, 42(3):597–618, 2014.

Paul A. Samuelson. Optimum social security in a life-cycle growth model. *International Economic Review*, 16(3):539–544, 1975.