

A model of financial contagion with variable asset returns may be replaced with a simple threshold model of cascades

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Abstract

I show the equivalence between a model of financial contagion and the widely-used threshold model of global cascades proposed by Watts (2002). The model financial network comprises banks that hold risky external assets as well as interbank assets. It turns out that there is no need to construct the balance sheets of banks if the shadow threshold of default is appropriately defined in accordance with the stochastic fluctuations in external assets.

Keywords: financial network, cascades, financial contagion, systemic risk.

JEL codes: G01, G18.

1 Introduction

Bank default is contagious. A failure of one bank can be spread through the financial network, generating default cascades. Over the past few years, many researchers in various fields of natural and social sciences, such as physicists, ecologists and economists, have been tackling the question of how to prevent financial contagion (e.g., Nier et al., 2007, Soramaki et al., 2007, Gai and Kapadia, 2010, Gai et al., 2011, Lenzu and Tedeschi, 2012, Kobayashi, 2013, Kobayashi and Hasui, 2013).

However, there is no wide agreement among researchers about how to construct a model of financial contagion. Different research groups use different models, which would make it difficult to establish a consensus about policy implications.¹

In this letter, I show the equivalence between a model of financial contagion and the widely-used threshold model of global cascades proposed by Watts (2002).² Basically, the financial network models require researchers to construct the banks' balance sheets. Then, the influence of a bank failure is examined by sequentially undermining the interbank assets of the lenders. Second-round defaults would occur if the fraction of defaulted borrowers among total borrowers exceeds a certain threshold.

These mechanisms are closely related to those of the Watts model of cascades. I find that there is no need to construct the balance sheets of banks as long as the threshold of default is appropriately defined in accordance with the fluctuations in asset values.

2 The models

2.1 A model of financial contagion

The model of financial contagion used in this paper is an extended version of the model used by Gai and Kapadia (2010). The only essential difference is that I take into account stochastic fluctuations in the value of external assets. A typical bank's balance sheet is illustrated in Figure 1.

¹See Lorenz et al. (2009) and Upper (2011) for a survey of the literature.

²See, for example, Dodds and Watts (2004), Gleeson and Cahalane (2007), Watts and Dodds (2007) and Gleeson (2013).

safe assets: b_i	deposits: d_i
interbank assets: $l_i = \sum_j \pi_{ji} \bar{p}_j$	
external risk assets: a_i	interbank liabilities: \bar{p}_i
	net worth: w_i

Fig. 1. A typical bank's balance sheet.

There are N banks in the financial market. Bank i holds risky external assets, a_i , interbank assets, l_i , and risk-less assets, b_i . In the liability side of the balance sheet, there are deposits, d_i , interbank liabilities, \bar{p}_i , and net worth, w_i . The balance-sheet condition implies that $a_i + l_i + b_i = d_i + \bar{p}_i + w_i$.

Banks are connected to each other by lending and borrowing. The existence of a lending-borrowing relationship is expressed as a *link* or an *edge*. In network theory, the number of outgoing links is called *out-degree*, while the number of incoming links is called *in-degree*. The direction of links represents the flow of funds at the time of initial lending.

The amount of bank i 's borrowings from bank j is expressed as $\pi_{ij} \bar{p}_i$, where π_{ij} denotes the relative weight of bank i 's borrowings from j , and thereby $\sum_{j \neq i} \pi_{ij} = 1$, $i = 1, \dots, N$. The amount of bank i 's total interbank assets is given by $l_i = \sum_{j \neq i}^N \pi_{ji} \bar{p}_j = k_i^{out} l_u$, where k_i^{out} , or out-degree, is the number of banks to which the bank lends, and l_u is the unit size of interbank lending.

Bank i will default if

$$\bar{p}_i > \sum_{j \neq i} \pi_{ji} p_j + \tilde{a}_i + b_i - d_i, \quad (1)$$

where \tilde{a}_i and p_j stand for the *ex-post* values of external assets and interbank liabilities, respectively. It should be pointed out that deposits, d_i , are reserved because deposits are

senior to interbank assets.

Provided that there is no loss in the interbank assets, banks will default with probability δ due to a loss of external assets. δ is called the probability of *fundamental defaults*, which is assumed to be common across banks. If bank j failed, then bank j 's creditors lose all of their credits extended to bank j . Some of these creditors may fail due to the loss of their interbank assets. Accordingly, the creditors of the creditors of bank j may fail as well, because they lose their credits extended to the failed banks.

Given the unit size of interbank lending, l_u , the sizes of interbank assets and liabilities are given by the structure of the interbank network. To ensure that the probability of fundamental defaults is the same across banks, the size of external assets relative to the net worth, θ_{aw} , is fixed. For those banks that have a positive amount of interbank assets, the ratio of total interbank assets to net worth, θ_{lw} , is also fixed. If bank i has so many incoming links that its liability side would be bigger than the sum of interbank assets and external assets, then the risk-less asset, b_i , is imposed to adjust the asset side. Otherwise, deposits, d_i , is imposed to adjust the liability side.

2.2 The Watts model of global cascades

Here, let us summarize the Watts' (2002) threshold model of cascades. In this model, each node in the network can take one of the two states, "flipped" or "not flipped". The network is undirected.

Let k and m denote the degree and the number of flipped neighbors, respectively. The algorithm of cascading behavior in the Watts model is that

$$\begin{cases} \text{node } i \text{ flips if } m_i > \phi k_i, \\ \text{node } i \text{ does not flip otherwise,} \end{cases}$$

where $\phi \in [0, 1]$ is the threshold of flipping. This means that the threshold number of flipped neighbor nodes above which node i will flip is $\lfloor \phi k_i \rfloor$, where $\lfloor x \rfloor$ denotes the floor function that returns the maximum integer smaller than x .

3 Equivalence between a financial network model and a threshold model

Here I show the equivalence between the financial network model and a modified version of the threshold model. Let us consider the financial network model. Recall that the value of external assets, a_i , may vary stochastically from bank to bank. Let $\Delta a_i \equiv \tilde{a}_i - a_i$ be the return of external assets. If $\Delta a_i = 0$, bank i will fail if the fraction of defaulted neighbors exceeds $\lfloor w_i/l_i \rfloor$. Notice that there is no possibility of contagious default if $w_i > l_i$, as long as there is no loss in the external assets.

If $\Delta a_i \neq 0$, on the other hand, the threshold of the fraction of defaulted neighbors depends on the realization of asset returns. If a bank gains positive (negative) returns from the external assets, then the bank becomes more resilient against (susceptible to) the failure of neighbor banks. Let $\tilde{\phi}_i$ be the “shadow” threshold of default for bank $i \in \{i \mid k_i^{out} > 0\}$. It follows that

$$\begin{aligned}\tilde{\phi}_i &= \frac{w_i + \Delta a_i}{l_i} \\ &= \frac{k_i^{out} l_u / \theta_{lw} + \Delta a_i}{k_i^{out} l_u} \\ &= \frac{1}{\theta_{lw}} + \frac{\Delta a_i}{k_i^{out} l_u}\end{aligned}\tag{2}$$

for $i \in \{i \mid k_i^{out} > 0\}$. Notice that the values of θ_{lw} , l_u and k_i^{out} are already given by the researcher. It turns out that if asset returns, Δa_i , follow a distribution of mean zero and variance σ^2 , then the shadow threshold $\tilde{\phi}_i$ follows a distribution of mean $\frac{1}{\theta_{lw}}$ and variance $\left(\frac{\sigma}{k_i^{out} l_u}\right)^2$. More generally, the p.d.f. of $\tilde{\phi}_i$, defined as $f_i(\cdot)$, is given by

$$f_i(r) = k_i^{out} l_u \cdot g_i(r \cdot k_i^{out} l_u - k_i l_u / \theta_{lw}), \quad i \in \{i \mid k_i^{out} > 0\}.\tag{3}$$

where $g_i(\cdot)$ is the p.d.f. of Δa_i .

Suppose, for instance, that asset returns follow a normal distribution with mean zero. Let $\tilde{z} \equiv F^{-1}(\delta)$, where F^{-1} is the inverse CDF of the standard normal distribution. The standard deviation of asset returns, σ_i , such that the probability of fundamental default

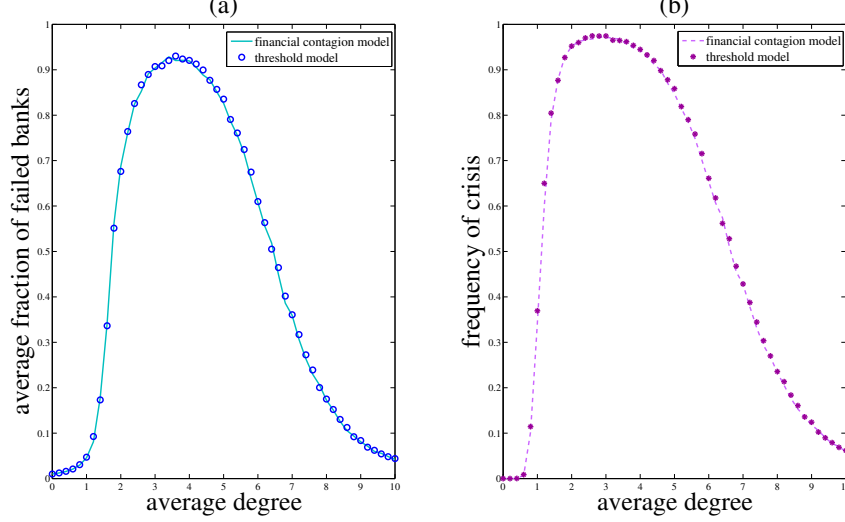


Fig. 2. Simulation results under a model of financial contagion and a threshold model.

Note: “Financial crisis” is defined as a situation in which at least 5% of banks go bankrupt.

becomes δ is given as

$$\begin{aligned}\sigma_i &= \frac{-w_i}{\tilde{z}}, \quad \forall i, \\ &= \frac{-k_i^{\text{out}} l_u}{\theta_{lw} \tilde{z}}, \quad \text{for } i \in \{i \mid k_i^{\text{out}} > 0\}.\end{aligned}\tag{4}$$

It follows that the shadow threshold follows a normal distribution with mean $1/\theta_{lw}$ and the standard deviation $1/(\theta_{lw} \tilde{z})$.

In this way, the model of financial contagion shown above can be expressed as a simple threshold model by introducing the concept of shadow threshold. Intuitively, the shadow threshold will become smaller as the returns of external assets take a lower value, meaning that the bank becomes more susceptible to default contagion. Those banks that have a negative value of ϕ_i will fail at the beginning, including the banks with no interbank assets.³ This corresponds to the case of fundamental defaults in the model of financial contagion.

Figure 2 shows the simulation results under both models. The parameter values are as

³Notice that $\text{Prob}(\tilde{\phi}_i < 0) = \text{Prob}(\frac{1}{\theta_{lw}} - \frac{x}{\theta_{lw} \tilde{z}} < 0) = F(x < \tilde{z}) = \delta$, where x is a random variable from the standard normal distribution.

follows: $N = 1000$, $\theta_{aw} = 7$, $\theta_{lw} = 3$, $l_u = 1$ and $\delta = .01$. These parameters suggest that the ratio of capital to total risky assets is .1 for those banks that have a positive amount of interbank assets and $1/7 \approx .143$ for the other banks. Given the average degree, Erdos-Rényi (directed) random networks are generated 20 times, and asset returns, or shadow thresholds, are generated 1000 times for each network structure.

4 Conclusion and discussion

In this letter, I showed that the introduction of shadow thresholds allows us to treat a model of financial contagion as a simple threshold model of cascades. There is no need to construct the balance sheets of banks. However, it is not clear whether this method can be applied to the other versions of the financial network models. For example, θ_{lw} and l_u are treated as constants in this model, but they generally vary across banks in practice. Another methodology will be needed in such more general cases.

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